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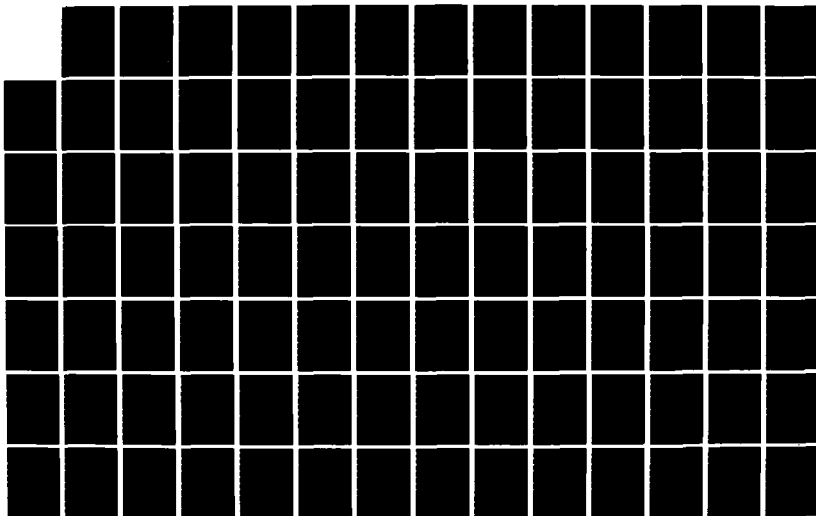
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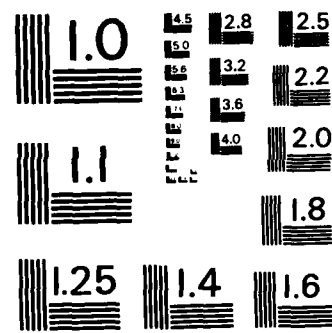
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COMPUTATION OF SPHERICAL HARMONICS  
AND APPROXIMATION BY SPHERICAL  
HARMONIC EXPANSIONS

WILLI FREEDEN

DEPARTMENT OF GEODETIC SCIENCE AND SURVEYING  
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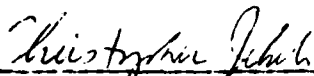
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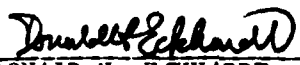
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A technique is developed for generating spherical harmonics by exact computation (in integer mode) thereby circumventing any source of rounding errors. Essential results of the theory of spherical harmonics are recapitulated by intrinsic properties of the space of homogeneous harmonic polynomials. Exact computation of (maximal) linearly independent and orthonormal systems of spherical harmonics is explained using exclusively integer operations. The numerical (con't)		

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efficiency is discussed.

The development of exterior gravitational potential in a series of outer (spherical) harmonics is investigated. Some numerical examples are given for solving exterior Dirichlet's boundary-value problems by use of outer (spherical) harmonic expansions for not-necessarily spherical boundaries. - 4 / 19

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# FOREWORD

This report was prepared by Dr. Willi Freeden, Associate Professor, Institute for Pure and Applied Mathematics, West Germany for the Department of Geodetic Science and Surveying, The Ohio State University, under Air Force Contract No. F19628-82-K-0017, Project Supervisor, Urho A. Uotila, Professor in the Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, Massachusetts, with Dr. Christopher Jekeli, Contract Manager.

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## Introduction

Spherical harmonics are closely associated with developments of gravitational or magnetic fields. Their main importance in geodesy has its roots in the (basis) property that the external gravitational potential of the earth may be approximated by suitable linear combinations of outer (spherical) harmonics.

Seen from potential theory the standard representation of a potential  $V$  defined outside a sphere about the origin with radius  $R$  by a series of outer harmonics

$$V(x) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=0}^n P_{nm}^* (\cos\theta) \{A_{nm} \cos(m\lambda) + B_{nm} \sin(m\lambda)\} \quad (0.1)$$

with

$r, \theta, \lambda$ : spherical coordinates

$P_{nm}^*$  : (normalized) Legendre function  
of degree  $n$  and order  $m$

$$P_{n0}^* (t) = \sqrt{2n+1} P_{n0} (t)$$

$$P_{nm}^* (t) = \sqrt{2(2n+1) \frac{(n-m)!}{(n+m)!}} P_{nm} (t) \quad (m > 0)$$

with

$$P_{nm} (t) = (1-t^2)^{m/2} \sum_{k=0}^{\left[\frac{n-m}{2}\right]} (-1)^k \frac{(2n-2k)!}{2^n k! (n-k)! (n-m-2k)!} t^{n-m-2k}$$

$A_{nm}, B_{nm}$  : potential coefficients of degree  $n$  and order  $m$

$$A_{n0} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi V(R, \theta, \lambda) P_{n0}^*(\cos\theta) \sin\theta d\theta d\lambda \quad (0.2)$$

$$\begin{Bmatrix} A_{nm} \\ B_{nm} \end{Bmatrix} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi V(R, \theta, \lambda) P_{nm}^*(\cos\theta) \begin{Bmatrix} \cos(m\lambda) \\ \sin(m\lambda) \end{Bmatrix} \sin\theta d\theta d\lambda$$

is well - considered as a series expansion by multipoles in the center of the sphere, i. e. the field of a homogeneous and isotropic distribution may be adequately approximated by developments of spherical harmonics. On the other hand, the series of spherical harmonics is less qualified for representing the field of a source distribution. Because of the slow convergence of the expansion comparatively laborious work must be done to approximate local (density) anomalies by a partial sum of the type

$$V_N(x) = \sum_{n=0}^N \left(\frac{R}{r}\right)^{n+1} \sum_{m=0}^n P_{nm}^*(\cos\theta) \{A_{nm} \cos(m\lambda) + B_{nm} \sin(m\lambda)\}. \quad (0.3)$$

The price to be paid is the choice of a relatively high degree  $N$ . This is why in the last decades many efforts have been done to reduce algorithmic diffi-

culties and computer time in generating spherical harmonics.

But not only the slow convergence for modelling local anomalies proves a difficulty in using spherical harmonic expansions. In practical applications, almost all activities to compute spherical harmonics numerically are restricted to the generation of one and only one system of representation (related to spherical coordinates), viz. the system

$$\left\{ \left( \frac{1}{r} \right)^{n+1} Y_{n,j}^* \right\} \quad \begin{matrix} n=0,1,\dots \\ j=1,\dots,2n+1 \end{matrix} \quad (0.4)$$

where  $Y_{n,j}^*$  is defined as follows

$$\begin{aligned} Y_{n,1}^* (\xi) &= P_{n0}^* (\cos \theta) \\ Y_{n,2}^* (\xi) &= P_{n1}^* (\cos \theta) \cos \lambda \\ Y_{n,3}^* (\xi) &= P_{n1}^* (\cos \theta) \sin \lambda \\ &\vdots \\ &\vdots \end{aligned} \quad (0.5)$$

$$\begin{aligned} Y_{n,2n}^* (\xi) &= P_{nn}^* (\cos \theta) \cos(n\lambda) \\ Y_{n,2n+1}^* (\xi) &= P_{nn}^* (\cos \theta) \sin(n\lambda) \end{aligned}$$

$$\left[ \begin{array}{l} x = r \xi, \quad \xi = (\sin \theta \cos \lambda, \sin \theta \sin \lambda, \cos \theta)^T \\ r = |x|, \quad 0 \leq \lambda < 2\pi, \quad 0 \leq \theta \leq \pi \end{array} \right].$$

Indeed, as it stands, the system (0.5) has many attractive features. Perhaps the most important are its simple structure and recursive computability.

A number of different equation sets has been described in the literature for suitably generating the system

$$\{Y_{n,j}^*\} \quad \begin{array}{l} n=0,1,\dots \\ j=1,\dots,2n+1 \end{array}$$

(e. g. Cl. Müller (1966), M. Gerstl (1978), C. Rizos (1979), O. Colombo (1982), R. H. Rapp (1982). A comparison of methods has been given by C. C. Tscherning, R. H. Rapp, C. Goad (1983)).

In doing so special care is always needed to ensure computational stability in the generation of the associated Legendre functions. It is a critical point to any calculation.

Furthermore, it should be mentioned that a potential  $V$  is related to polar coordinates when choosing an approximation of type (0.3). Thus differentiations with respect to cartesian coordinates cannot be performed in a straightforward manner. In order to determine the gravitational part of the force acting on an artificial satellite close to earth, for example, we need essentially the gradient of the external gravitational potential of the earth. Its representation must be free of singularities caused by a special choice of a coordinate system.

This is done in canonical manner by use of spherical harmonic representations with respect to cartesian coordinates. In addition, cartesian coordinates seem to be more adequate if the attempt is made to approximate the gravitational part of the earth's gravity potential by means of outer harmonics (multipoles) using a non-spherical earth's model (e. g. ellipsoid, spheroid, telluroid) as proposed by the author (1983).

Thus the question arises of developing appropriate algorithms for numerical computation of spherical harmonics related to cartesian coordinates thereby avoiding any computational problem of accuracy and stability.

The purpose of this report is to demonstrate both that spherical harmonics can be evaluated exactly using exclusively integer operations and that the procedure of expanding the external gravitational potential of the earth into spherical harmonics can be performed adequately by use of outer harmonics in terms of cartesian coordinates.

The report is meant to be a proposal for exact computation of spherical harmonics; it is not understood to be the final word in speed and economy in computations of e. g. high degree harmonics. Nevertheless, at least in the opinion of the author, it is a remarkable discovery that, to any

prescribed degree  $n$ , orthonormal systems of spherical harmonics can be deduced exactly, i. e. exclusively by addition, subtraction and multiplication of integers, without having any knowledge of spherical harmonics of orders different from  $n$ . Hence, as long as we (are able to) use integer operations, problems of accuracy and stability do not occur; the algorithm depends merely on available computer time and memory.

Moreover, our approach provides us with exactly given linearly independent systems of homogeneous harmonic polynomials (in terms of cartesian coordinates) which can be used for developments of the earth's gravitational potential in series of outer harmonics (adapted to a non-spherical earth's model).

In detail we are concerned with the following considerations:

#### Exact Computation of Spherical Harmonics:

Starting point is the class  $P_n$  of homogeneous harmonic polynomials of degree  $n$ . Each  $H_n \in P_n$  may be represented in the form

$$\begin{aligned} H_n(x) &= \sum_{[\alpha]=n} C_\alpha x^\alpha & (0.6) \\ &= \sum_{\alpha_1+\alpha_2+\alpha_3=n} C_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \end{aligned}$$

( $\alpha$ : multiindex of degree  $n$ ). It is obvious that the set of monomials

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \quad [\alpha] = n, \quad (0.7)$$

forms a basis for the space  $P_n$ . The number of such monomials is precisely equal to

$$M = M(n) = \binom{n+2}{2}. \quad (0.8)$$

$H_n$  denotes the class of polynomials in  $P_n$  that are harmonic

$$H_n = \{H_n \in P_n \mid \Delta_x H_n(x) = 0\}. \quad (0.9)$$

The dimension of  $H_n$  is  $2n+1$ . In other words, there are  $2n+1$  linearly independent functions  $H_{n,j}$  in  $H_n$ :

$$H_{n,j}(x) = \sum_{[\alpha]=n} C_\alpha^j x^\alpha. \quad (0.10)$$

( $j=1, \dots, 2n+1$ )

The main result of the paper is that the coefficients  $C_\alpha^j$  can be obtained as integers by integer operations and that the solution process of determining the matrix  $c$  consisting of the column vectors

$$(C_\alpha^1), \dots, (C_\alpha^{2n+1})$$

can be made surprisingly simple and reasonably efficient.

In view of the homogeneity the functions  $H_{n,j}$  may be rewritten as follows

$$H_{n,j}(x) = |x|^n \sum_{[\alpha]=n} C_{\alpha}^j \xi^{\alpha} .$$

$$(x = |x|\xi , \quad |\xi| = 1)$$

The functions

$$H_{n,j}(\xi) = \sum_{[\alpha]=n} C_{\alpha}^j \xi^{\alpha} , \quad (0.11)$$

$$(j = 1, \dots, 2n+1)$$

form a maximal linearly independent system in the space  $S_n$  of all (surface) spherical harmonics of degree  $n$ . Consequently, the system

$$\left\{ \frac{1}{|x|^{2n+1}} H_{n,j}(x) \right\}_{j=1, \dots, 2n+1} (|x| > 0) \quad (0.12)$$

represents a maximal linearly independent system of outer (spherical) harmonics of order  $n$ . Obviously, the system (0.12) does not consist of polynomials but it is a set of algebraic functions which are homogeneous of degree  $-(n+1)$ .

Corresponding to the maximal linearly independent system (0.11) of surface spherical harmonics of order  $n$  there exists a system

$$\{S_{n,j}^*\}_{j=1, \dots, 2n+1} \quad (0.13)$$



such that

$$\frac{1}{4\pi} \int_{|\xi|=1} S_{n,j}^*(\xi) S_{n,l}^*(\xi) d\omega = \delta_{jl}$$

( $d\omega$ : surface element,  $\delta_{jl}$ : Kronecker symbol).

As usual the system can be constructed by linear combination of the members of the system (0.11), i. e. each  $S_{n,j}^*$  may be represented as follows

$$S_{n,j}^*(\xi) = \sum_{l=1}^{2n+1} b_{n,l}^j H_{n,l}(\xi), \quad |\xi| = 1. \quad (0.14)$$

But this means that we can find  $2n+1$  vectors

$$(B_{\alpha}^1), \dots, (B_{\alpha}^{2n+1})$$

determined by

$$B_{\alpha}^j = \sum_{l=1}^{2n+1} b_{n,l}^j C_{\alpha}^l, \quad [\alpha] = n,$$

such that

$$S_{n,j}^*(\xi) = \sum_{[\alpha]=n} B_{\alpha}^j \xi^{\alpha}, \quad |\xi| = 1. \quad (0.15)$$

It will be shown that the coefficients  $B_{\alpha}^j$  can be determined again only by application of integer operations.

**Approximation by series expansion:**

Any function  $V$  satisfying the following properties

- (i)  $V$  is continuous in  $|x| \geq R$  and twice continuously differentiable in  $|x| > R$
- (ii)  $V$  is harmonic in  $|x| > R$
- (iii)  $V$  is regular at infinity

may be represented in the form

$$V(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\frac{R}{|x|}\right)^{n+1} \left\{ \frac{1}{4\pi} \int_{|\eta|=1} V(R\eta) S_{n,j}^*(\eta) d\omega \right\} S_{n,j}^*(\xi). \quad (0.16)$$

$$(x = r\xi, \quad r = |x|, \quad |\xi| = 1)$$

More explicitly this reads: given an error bound  $\varepsilon > 0$ , then there exists an integer  $N = N(\varepsilon)$  such that

$$\left| V(x) - \sum_{n=0}^N \sum_{j=1}^{2n+1} \left(\frac{R}{|x|}\right)^{n+1} \left\{ \frac{1}{4\pi} \int_{|\eta|=1} V(R\eta) S_{n,j}^*(\eta) d\omega \right\} S_{n,j}^*(\xi) \right| \leq \varepsilon \quad (0.17)$$

holds for all points  $x \in R^3$  with  $|x| \geq R_0 > R$ .

From the addition theorem (decomposition formula) it follows that

$$\begin{aligned} \sqrt{2n+1} P_{no}^* (\xi \eta) &= \sum_{j=1}^{2n+1} Y_{n,j}^* (\xi) Y_{n,j}^* (\eta) \quad (0.18) \\ &= \sum_{j=1}^{2n+1} S_{n,j}^* (\xi) S_{n,j}^* (\eta) \end{aligned}$$

for any two unit vectors  $\xi, \eta$ . Therefore, for every index  $N$  and all  $x \in R^3$  with  $|x| \geq R_0 > R$  our approximation  $V_N$  to  $V$

$$V_N(x) = \sum_{n=0}^N \sum_{j=1}^{2n+1} \left(\frac{R}{|x|}\right)^{n+1} \left\{ \frac{1}{4\pi} \int_{|\eta|=1} V(R\eta) S_{n,j}^*(\eta) d\omega \right\} S_{n,j}^*(\xi) \quad (0.19)$$

coincides with the (standard) approximation

$$V_N(x) = \sum_{n=0}^N \sum_{j=1}^{2n+1} \left(\frac{R}{|x|}\right)^{n+1} \left\{ \frac{1}{4\pi} \int_{|\eta|=1} V(R\eta) Y_{n,j}^*(\eta) d\omega \right\} Y_{n,j}^*(\xi). \quad (0.20)$$

Furthermore, each element  $Y_{n,j}^*$  is expressible in terms of the system  $\{S_{n,j}^*\}_{j=1, \dots, 2n+1}$ , and vice versa. In other words, even if we base our approximation process on the conventional system

$\{Y_{n,j}^*\}_{n=0,1,\dots, j=1,\dots,2n+1}$  we are able to do this by means

of the functions  $S_{n,j}^*$ , i. e. suitable linear combinations of the system (0.11).

The paper ends with some reflections about the role of outer harmonics (multipoles) in orthogonal expansions using a non-spherical earth's model. The mathematical procedure and the theoretical background developed by the author (1983) are briefly recapitulated. Many advantages of the classical (strictly) spherical approach, of course, cannot be maintained, when outer harmonics will be used as trial functions in approximations of the external gravitational potential adapted to a non-spherical earth's model (e. g. ellipsoid, geoid, telluroid, spheroid, and (at least in principle) real earth). Nevertheless, numerical examples make us hope to implement an outer harmonic (multipole) approach to the external gravitational potential of the earth related to a non-spherical earth's surface.

Seen from mathematical point of view, the sample examples open at the same time new numerical perspectives in solving exterior Dirichlet's boundary-value problems by orthogonal expansion using outer harmonics.

## 1. Notations

$R^3$  denotes three dimensional (real) Euclidean space. We write  $x, y, \dots$  to represent the elements of  $R^3$ . In components we have

$$x = (x_1, x_2, x_3)^T, \quad y = (y_1, y_2, y_3)^T. \quad (1.1)$$

Scalar product and norm are defined, as usual

$$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 \quad (1.2)$$

$$x^2 = x \cdot x = x_1^2 + x_2^2 + x_3^2 \quad (1.3)$$

$$|x| = \sqrt{x^2} = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad (1.4)$$

Let  $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$  be a triple of non-negative integers  $\alpha_1, \alpha_2, \alpha_3$ . We set

$$\alpha! = \alpha_1! \alpha_2! \alpha_3! \quad (1.5)$$

$$[\alpha] = \alpha_1 + \alpha_2 + \alpha_3. \quad (1.6)$$

We say  $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$  is a multiindex of degree  $n$  if  $[\alpha] = n$ ,  $n$ : non-negative integer.

As usual, we set

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

and (1.7)

$$(\nabla_x)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3} = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} .$$

An easy calculation gives

$$(x_1 + x_2 + x_3)^n = \sum_{[\alpha]=n} \frac{n!}{\alpha!} x^\alpha$$

and (1.8)

$$(x_1 y_1 + x_2 y_2 + x_3 y_3)^n = \sum_{[\alpha]=n} \frac{n!}{\alpha!} x^\alpha y^\alpha .$$

Furthermore, we have

$$(\nabla_x)^\alpha x^\beta = \begin{cases} 0 & \text{for } \alpha \neq \beta \text{ and } [\alpha] = [\beta] \\ \alpha! & \text{for } \alpha = \beta . \end{cases} \quad (1.9)$$

Each  $x \in \mathbb{R}^3$ ,  $x = (x_1, x_2, x_3)^T$  with  $|x| \neq 0$ , admits a representation of the form

$$x = r\xi, \quad r = |x|, \quad \xi = (\xi_1, \xi_2, \xi_3)^T, \quad (1.10)$$

where  $\xi \in \mathbb{R}^3$ ,  $|\xi| = 1$ , is the uniquely determined directional (unit) vector of  $x$ . The unit sphere in  $\mathbb{R}^3$  will be called  $\Omega$ . As it is well-known, the total surface  $\|\Omega\|$  of  $\Omega$  is equal to  $4\pi$ :

$$\|\Omega\| = \int_{\Omega} d\omega = 4\pi \quad (d\omega: \text{ surface-element}),$$

Let  $\epsilon^1, \epsilon^2, \epsilon^3$  form the (canonical) orthonormal basis in  $R^3$ :

$$\epsilon^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then we may represent the points  $\xi \in \Omega$  in polar coordinates

$$\xi = t \epsilon^3 + \sqrt{1-t^2} (\cos \lambda \epsilon^1 + \sin \lambda \epsilon^2) \quad (1.11)$$

$$-1 \leq t \leq +1, \quad 0 \leq \lambda < 2\pi, \quad t = \cos \theta,$$

( $\theta$ : polar distance,  $\lambda$ : geocentric longitude):

$$\xi^T = (\sin \theta \cos \lambda, \sin \theta \sin \lambda, \cos \theta)^T.$$

In terms of polar coordinates the Laplace-operator  $\Delta$  in  $R^3$

$$\Delta_x = \nabla_x \cdot \nabla_x = \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2 \quad (1.12)$$

is represented as follows

$$\Delta_x = \left(\frac{\partial}{\partial r}\right)^2 + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\xi}^*, \quad (1.13)$$

where  $\Delta^*$  denotes the (Laplace-) Beltrami operator of the unit sphere  $\Omega$

$$\Delta_{\xi}^* = (1-t^2) \left(\frac{\partial}{\partial t}\right)^2 - 2t \frac{\partial}{\partial t} + \frac{1}{1-t^2} \left(\frac{\partial}{\partial \lambda}\right)^2. \quad (1.14)$$

## 2. Homogeneous Polynomials

Let  $P_n$  be the set of all homogeneous polynomials of degree  $n$  on  $R^3$ . Thus, if  $H_n \in P_n$ , then

$$\begin{aligned} H_n(x) &= \sum_{\alpha_1 + \alpha_2 + \alpha_3 = n} C_{\alpha_1 \alpha_2 \alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \\ &= \sum_{[\alpha]=n} C_{\alpha} x^{\alpha}. \end{aligned} \quad (2.1)$$

It is obvious that the set of monomials  $x^{\alpha}$ ,  $[\alpha]=n$ , is a basis for the space  $P_n$ . The number of such monomials is precisely the number,  $M = M(n)$ , of ways a triple  $\alpha$  of non-negative integers can be chosen so that we have  $[\alpha]=n$ . Thus  $M$  is equal to the number of ways of selecting 2 elements out of a collection of  $n+2$  ones. This means, the dimension of  $P_n$  is equal to

$$M = M(n) = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}. \quad (2.2)$$

Let  $H_n(\nabla_x)$  be the differential operator associated to  $H_n(x)$  (i. e.: replace  $x^{\alpha}$  formally by  $\nabla_x^{\alpha}$  in the expression of  $H_n(x)$ ):

$$\begin{aligned} H_n(\nabla_x) &= \sum_{[\alpha]=n} C_{\alpha_1 \alpha_2 \alpha_3} \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \\ &= \sum_{[\alpha]=n} C_{\alpha} (\nabla_x)^{\alpha}. \end{aligned} \quad (2.3)$$



If such an operator is applied on a homogeneous polynomial  $L_n$  of the same degree  $n$

$$L_n(x) = \sum_{[\beta]=n} D_\beta x^\beta \quad (2.4)$$

we obtain as result a real number:

$$\begin{aligned} [H_n(\nabla_x)] L_n(x) &= \sum_{[\alpha]=n} \sum_{[\beta]=n} C_\alpha D_\beta \left[ \left( \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} x_1^{\beta_1} \right) \left( \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} x_2^{\beta_2} \right) \left( \frac{\partial^{\alpha_3}}{\partial x_3^{\alpha_3}} x_3^{\beta_3} \right) \right] \\ &= \sum_{[\alpha]=n} C_\alpha D_\alpha \alpha! . \end{aligned} \quad (2.5)$$

Clearly, we find

$$\begin{aligned} [H_n(\nabla_x)] L_n(x) &= [L_n(\nabla_x)] H_n(x), \\ [H_n(\nabla_x)] H_n(x) &\geq 0 . \end{aligned} \quad (2.6)$$

This gives rise to introduce an inner product  $(\cdot, \cdot)_{P_n}$  on the space  $P_n$  by letting

$$(H_n, L_n)_{P_n} = [H_n(\nabla_x)] L_n(x) \quad (2.7)$$

for  $H_n$  and  $L_n$  homogeneous polynomials of degree  $n$ .  
Clearly, we have

$$(i) \quad (c H_n, L_n)_{P_n} = c (H_n, L_n)_{P_n} \quad \text{for } c \in \mathbb{R}$$

$$(ii) \quad (H_n + K_n, L_n)_{P_n} = (H_n, L_n)_{P_n} + (K_n, L_n)_{P_n}$$

for  $H_n, L_n, K_n \in P_n$

$$(iii) \quad (H_n, L_n)_{P_n} = (L_n, H_n)_{P_n}$$

$$(iv) \quad (H_n, H_n)_{P_n} > 0 \quad \text{for } H_n \neq 0.$$

The space  $P_n$  equipped with the inner product  $(\cdot, \cdot)_{P_n}$  is a finite-dimensional Hilbert space. The set of  $n$  monomials

$$\left\{ \sqrt{\frac{1}{\alpha!}} x^\alpha \mid [\alpha] = n \right\} \quad (2.8)$$

forms a complete and closed system of orthonormal elements in  $P_n$ .

For each  $H_n \in P_n$  we have

$$\begin{aligned} H_n(x) &= \sum_{[\alpha]=n} \frac{1}{\alpha!} [H_n(\nabla_y)] y^\alpha x^\alpha & (2.9) \\ &= [H_n(\nabla_y)] \frac{1}{n!} \sum_{[\alpha]=n} \frac{n!}{\alpha!} x^\alpha y^\alpha \\ &= [H_n(\nabla_y)] \frac{(x \cdot y)^n}{n!} \\ &= \frac{1}{n!} (x \cdot \nabla_y)^n H_n(y) . \end{aligned}$$

In other words,

$$H_n(x) = \left( \frac{(xy)^n}{n!}, H_n(y) \right)_{P_n}. \quad (2.10)$$

$P_n$  equipped with the inner product  $(\cdot, \cdot)_{P_n}$  is a  $M(n)$ -dimensional Hilbert space with the reproducing kernel

$$K(n; x, y) = \frac{(x \cdot y)^n}{n!}. \quad (2.11)$$

i. e.: (i) For every fixed  $y$ , the function  $K(n; \cdot, y)$  belongs to  $P_n$ .

(ii) For any  $H_n \in P_n$  and any point  $x$  the reproducing property

$$H_n(x) = (K(n; x, y), H_n(y))_{P_n}$$

is valid.

$K(n; x, y)$  is the only reproducing kernel in  $P_n$ .

Now, let  $\{H_{n,j}\}_{j=1,\dots,M}$ ,  $\{L_{n,j}\}_{j=1,\dots,M}$  be two orthonormal systems in  $P_n$ :

$$(H_{n,j}, H_{n,k})_{P_n} = \delta_{jk} \quad (2.12)$$

$$(L_{n,j}, L_{n,k})_{P_n} = \delta_{jk}.$$

Then, for  $j = 1, \dots, M$ , we have

$$H_{n,j} = \sum_{k=1}^M L_{n,k} (H_{n,j}, L_{n,k}) P_n \quad (2.13)$$

$$L_{n,j} = \sum_{k=1}^M H_{n,k} (L_{n,j}, H_{n,k}) P_n .$$

Therefore it follows that

$$\sum_{j=1}^M H_{n,j}(x) H_{n,j}(y) = \sum_{j=1}^M \sum_{k=1}^M L_{n,k}(x) H_{n,j}(y) (H_{n,j}, L_{n,k}) P_n \quad (2.14)$$

$$\sum_{j=1}^M L_{n,j}(x) L_{n,j}(y) = \sum_{j=1}^M \sum_{k=1}^M L_{n,j}(x) H_{n,k}(y) (L_{n,j}, H_{n,k}) P_n .$$

Consequently,

$$\sum_{j=1}^M H_{n,j}(x) H_{n,j}(y) = \sum_{j=1}^M L_{n,j}(x) L_{n,j}(y) . \quad (2.15)$$

Hence, in particular for the orthonormal system of monomials (2.8), we obtain

Let  $\{H_{n,j}\}_{j=1,\dots,M}$  be an orthonormal system in  $P_n$ . Then

$$\frac{(x \cdot y)^n}{n!} = \sum_{j=1}^M H_{n,j}(x) H_{n,j}(y) .$$

Given  $M$  points  $x_1, \dots, x_M \in R^3$  and  $M$  values  $d_1, \dots, d_M \in R$ , we will be able to solve the interpolation problem

$$\sum_{k=1}^M b_k H_{n,k}(x_j) = d_j, \quad j = 1, \dots, M \quad (2.16)$$

if and only if

$$\det (H_{n,k}(x_j))_{\substack{j=1,\dots,M \\ k=1,\dots,M}} \neq 0 . \quad (2.17)$$

A system of  $M$  points  $x_1, \dots, x_M$  is called **fundamental system** relative to  $P_n$  if the matrix

$$(H_{n,k}(x_j))_{\substack{j=1,\dots,M \\ k=1,\dots,M}} \quad (2.18)$$

is of maximal rank  $M$ .

Let us prove the existence of a fundamental system relative to  $P_n$ . As orthonormal system  $\{H_{n,j}\}$ , the functions  $H_{n,1}, \dots, H_{n,M}$  are linearly independent. Hence, there exists a point  $x_1$  for which

$$H_{n,1}(x_1) \neq 0.$$

Now, there must also be a point  $x_2$  such that

$$\begin{vmatrix} H_{n,1}(x_1) & H_{n,1}(x_2) \\ H_{n,2}(x_1) & H_{n,2}(x_2) \end{vmatrix} \neq 0$$

for else we would have a contradiction to the linear independence of  $H_{n,1}$ ,  $H_{n,2}$ . In the same way the existence of a point  $x_3$  can be deduced by the requirement

$$\begin{vmatrix} H_{n,1}(x_1) & H_{n,1}(x_2) & H_{n,1}(x_3) \\ H_{n,2}(x_1) & H_{n,2}(x_2) & H_{n,2}(x_3) \\ H_{n,3}(x_1) & H_{n,3}(x_2) & H_{n,3}(x_3) \end{vmatrix} \neq 0$$

Finally, we obtain a system of points  $x_1, \dots, x_M$  such that

$$\begin{vmatrix} H_{n,1}(x_1) & \dots & H_{n,1}(x_M) \\ \vdots & & \vdots \\ H_{n,M}(x_1) & \dots & H_{n,M}(x_M) \end{vmatrix} \neq 0$$

i. e. a fundamental system relative to  $P_n$ .

To every  $H_n \in P_n$ , there exist real numbers  $b_1, \dots, b_M$  such that

$$H_n = \sum_{k=1}^M b_k H_{n,k} \quad (2.19)$$

Now, assumed  $x_1, \dots, x_M$  is a fundamental system relative to  $P_n$ , the linear equations

$$\sum_{j=1}^M a_j H_{n,j}(x_k) = b_k, \quad k=1, \dots, M, \quad (2.20)$$

are uniquely solvable in the unknowns  $a_1, \dots, a_M$ .

Thus we obtain

$$H_n = \sum_{k=1}^M \sum_{j=1}^M a_j H_{n,j}(x_k) H_{n,j} \quad (2.21)$$

Let  $\{H_{n,j}\}_{j=1, \dots, M}$  be an orthonormal system in  $P_n$ . Assume that  $\{x_k\}_{k=1, \dots, M}$  is a fundamental system relative to  $P_n$ . Then, each  $H_n \in P_n$  is uniquely representable in the form

$$H_n(x) = \sum_{j=1}^M a_j K(n; x_j, x) .$$

### 3. Homogeneous Harmonic Polynomials

Let  $H_n \subset P_n$  be the class of all polynomials in  $P_n$  that are harmonic:

$$H_n = \{H_n \in P_n \mid \Delta_x H_n(x) = 0\} . \quad (3.1)$$

For  $n < 2$ , all homogeneous polynomials are harmonic.

For  $n \geq 2$ , let  $H_{n-2}$  be a homogeneous polynomial of degree  $n-2$ , i. e.  $H_{n-2} \in P_{n-2}$ . Then, for each homogeneous harmonic polynomial  $K_n$ , we have

$$\begin{aligned} & (|x|^2 H_{n-2}(x) , K_n(x))_{P_n} \\ &= [H_{n-2} (\nabla_x)] \Delta_x K_n(x) \\ &= 0 . \end{aligned} \quad (3.2)$$

This means  $|x|^2 H_{n-2}(x)$  is orthogonal to  $K_n(x)$  in the sense of the inner product  $(\cdot, \cdot)_{P_n}$ .

Conversely, suppose that  $H_n$  is orthogonal to all elements  $L_n$  of the form

$$L_n(x) = |x|^2 H_{n-2}(x) , \quad H_{n-2} \in P_{n-2} ,$$

then it follows that



$$\begin{aligned}
 0 &= (|x|^2 H_{n-2}(x), H_n(x))_{P_n} & (3.3) \\
 &= [H_{n-2}(\nabla_x)] \Delta_x H_n(x) \\
 &= (H_{n-2}, \Delta H_n)_{P_{n-2}}
 \end{aligned}$$

for all  $H_{n-2} \in P_{n-2}$ . This is true only if  $\Delta H_n = 0$ , i. e.  $H_n$  is a homogeneous harmonic polynomial.

$P_n$ ,  $n \geq 2$ , is the orthogonal direct sum of  $H_n$  and  $H_n^\perp$ , where

$$H_n^\perp = |x|^2 P_{n-2}$$

is the space of all members  $L_n$  with  $L_n(x) = |x|^2 L_{n-2}(x)$ ,  $L_{n-2} \in P_{n-2}$ , i. e.: each homogeneous polynomial  $H_n$  of degree  $n$  can be decomposed uniquely in the form

$$H_n(x) = K_n(x) + |x|^2 H_{n-2}(x),$$

where  $K_n$  is a homogeneous harmonic polynomial of degree  $n$  and  $H_{n-2}$  is a homogeneous polynomial of degree  $n-2$ .

Denote by  $p$  and  $p_\perp$  the projection operators in  $P_n$  onto  $H_n$  and  $H_n^\perp$ , respectively. Then,

$$H_n = p H_n + p_\perp H_n. \quad (3.4)$$

In other words,

$$K_n(x) = p H_n(x) \quad (3.5)$$

$$|x|^2 H_{n-2}(x) = p_{\perp} H_n(x) .$$

Clearly,

$$p^2 = p . \quad (3.6)$$

For all  $H_n, L_n \in P_n$

$$(pH_n, L_n)_{P_n} = (H_n, pL_n)_{P_n} . \quad (3.7)$$

Moreover, we have

$$pH_n = pK_n = K_n . \quad (3.8)$$

The dimension  $N(n)$  of  $H_n$  follows easily from the fact that

$$\begin{aligned} N(n) &= \dim H_n \\ &= \dim P_n - \dim H_{\perp} \\ &= \dim P_n - \dim P_{n-2} . \end{aligned} \quad (3.9)$$

Explicitly, this reads

$$\begin{aligned} \dim H_n &= \binom{n+2}{2} - \binom{n}{2} \\ &= 2n + 1 . \end{aligned} \quad (3.10)$$

This means there exist  $(2n+1)$ -linearly independent homogeneous harmonic polynomials of degree  $n$ .

#### 4. Addition Theorem

Our aim now is to give the explicit representation of the orthogonal projection  $pH_n$  of a given homogeneous polynomial  $H_n$ . For that purpose we need some preliminaries.

A result due to E. W. Hobson (1955) yields for  $i = 1, 2, 3$

$$\left(\frac{\partial}{\partial x_i}\right)^n \frac{1}{|x|} = (-1)^n \frac{(2n)!}{(n!) 2^n} \frac{1}{|x|^{2n+1}} \left[ \sum_{s=0}^{[n/2]} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s \right] x_i^n,$$

where we have set

$$[n/2] = \frac{1}{2} (n - \frac{1}{2} [1 - (-1)^n]) ,$$

i. e.:

$$[n/2] = \begin{cases} \frac{n}{2} & \text{for } n \text{ even} \\ \frac{n-1}{2} & \text{for } n \text{ odd} . \end{cases}$$

In other words, we find

$$\left(\epsilon^i \nabla_x\right)^n \frac{1}{|x|} = (-1)^n \frac{(2n)!}{(n!) 2^n} \frac{1}{|x|^{2n+1}} \left[ \sum_{s=0}^{[n/2]} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s \right] (\epsilon^i x)^n.$$

(i = 1, 2, 3)

Since the differential operator  $\Delta$  is invariant with respect to rotations, it is easy to see that

$$(\nabla_x)^n \frac{1}{|x|} = (-1)^n \frac{(2n)!}{(n!) 2^n} \frac{1}{|x|^{2n+1}} \left[ \sum_{s=0}^{[n/2]} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s \right] (yx)^n$$

is valid for every  $y \in R^3$ . Now, as we have seen in chapter 2, each  $H_n \in P_n$  may be represented in the form

$$H_n(x) = \sum_{j=1}^M c_j (x_j \cdot x)^n,$$

where  $c_j$ ,  $j=1, \dots, M$ , are suitable coefficients and  $\{x_j\}_{j=1, \dots, M}$  is a fundamental system relative to  $P_n$ .

Consequently, we have

Let  $H_n$  be a homogeneous polynomial of degree  $n$ . Then for  $x \in R^3$ ,  $|x| \neq 0$

$$[H_n (\nabla_x)] \frac{1}{|x|} \tag{4.1}$$

$$= (-1)^n \frac{(2n)!}{2^n (n!)} \frac{1}{|x|^{2n+1}} \left[ \sum_{s=0}^{[n/2]} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s \right] H_n(x) .$$

From the considerations given in chapter 3 it follows that

$$[H_n(\nabla_x)] \frac{1}{|x|} = [K_n(\nabla_x)] \frac{1}{|x|} + [H_{n-2}(\nabla_x)] \Delta_x \frac{1}{|x|} . \quad (4.2)$$

Thus, in connection with

$$\begin{aligned} \Delta_x \frac{1}{|x|} &= 0, & |x| &\neq 0 \\ \Delta_x K_n(x) &= 0, & x &\in \mathbb{R}^3, \end{aligned} \quad (4.3)$$

we obtain for  $|x| \neq 0$

$$\begin{aligned} [H_n(\nabla_x)] \frac{1}{|x|} &= [K_n(\nabla_x)] \frac{1}{|x|} \\ &= (-1)^n \frac{(2n)!}{(n!) 2^n} \frac{1}{|x|^{2n+1}} K_n(x) . \end{aligned} \quad (4.4)$$

Therefore, by comparison of (4.1) and (4.4), we get

Let  $H_n$  be a homogeneous polynomial of degree  $n$ .  
Then

$$pH_n(x) = \sum_{s=0}^{[n/2]} (-1)^s \frac{n!(2n-2s)!}{(2n)!(n-s)!s!} |x|^{2s} \Delta^s H_n(x) . \quad (4.5)$$

Observing

$$\Delta_x (xy)^n = n(n-1) |y|^2 (xy)^{n-2}, \quad y \in \mathbb{R}^3, \quad (4.6)$$

we obtain, in particular,

$$\begin{aligned} p \left( \frac{(xy)^n}{n!} \right) & \quad (4.7) \\ &= \frac{1}{n!} \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!(n!)^2}{(n-2s)!(n-s)!s!(2n)!} |x|^{2s} |y|^{2s} (xy)^{n-2s}. \end{aligned}$$

Hence, after an easy calculation, we find by using polar coordinates

$$\begin{aligned} x &= |x| \xi, & \xi &\in \Omega \\ y &= |y| \eta, & \eta &\in \Omega \end{aligned}$$

the equation

$$\begin{aligned} p \left( \frac{(xy)^n}{n!} \right) & \quad (4.8) \\ &= \frac{(2n+1) 2^n \cdot n!}{(2n+1)!} (|x| |y|)^n \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n (n-2s)!(n-s)!s!} (\xi\eta)^{n-2s} \\ &= \frac{2n+1}{1 \cdot 3 \cdot \dots \cdot 2n+1} |x|^n |y|^n \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n (n-2s)!(n-s)!s!} (\xi\eta)^{n-2s}. \end{aligned}$$

Let  $\{H_{n,j}\}_{j=1,\dots,2n+1}$  be a maximal orthonormal system in  $H_n$  and  $\{L_{n,j}\}_{j=1,\dots,M-(2n+1)}$  be a maximal orthonormal system in  $H_n^\perp$ . Then the union of both systems

$$\{H_{n,j}\}_{j=1,\dots,2n+1} \cup \{L_{n,j}\}_{j=1,\dots,M-(2n+1)} \quad (4.9)$$

forms a maximal orthonormal system in  $P_n$ .

Therefore it follows that

$$\begin{aligned} & \frac{(xy)^n}{n!} \\ &= \sum_{j=1}^{2n+1} H_{n,j}(x) H_{n,j}(y) + \sum_{j=1}^{M-(2n+1)} L_{n,j}(x) L_{n,j}(y) \end{aligned} \quad (4.10)$$

for any two elements  $x, y \in \mathbb{R}^3$ .

Furthermore, in view of the definition of the projection operator  $p$ , we get

$$\begin{aligned} & p \left( \sum_{j=1}^{2n+1} H_{n,j}(x) H_{n,j}(y) + \sum_{j=1}^{M-(2n+1)} L_{n,j}(x) L_{n,j}(y) \right) \\ &= p \left( \sum_{j=1}^{2n+1} H_{n,j}(x) H_{n,j}(y) \right) + p \left( \sum_{j=1}^{M-(2n+1)} L_{n,j}(x) L_{n,j}(y) \right) \\ &= \sum_{j=1}^{2n+1} H_{n,j}(x) H_{n,j}(y) . \end{aligned} \quad (4.11)$$

On the other hand, as we have shown above

$$p \left( \frac{(x \cdot y)^n}{n!} \right) \quad (4.12)$$

$$= \frac{(2n+1) 2^n n!}{(2n+1)!} |x|^n |y|^n \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n (n-2s)! (n-s)! s!} (\xi \eta)^{n-2s} .$$



By comparison of (4.11) and (4.12) this yields the addition theorem of homogeneous harmonic polynomials:

Let  $\{H_{n,j}\}_{j=1,\dots,2n+1}$  be an orthonormal system in  $H_n$  with respect to  $(\cdot, \cdot)_n$ . Then, for  $x, y \in \mathbb{R}^3$ , we have

$$\begin{aligned} & \sum_{j=1}^{2n+1} H_{n,j}(x) H_{n,j}(y) \\ &= \frac{(2n+1) 2^n n!}{(2n+1)!} |x|^n |y|^n P_n(\xi\eta), \end{aligned}$$

$$(x = |x| \xi, \quad y = |y| \eta; \quad \xi, \eta \in \Omega)$$

where we have used the abbreviation

$$P_n(t) = \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n (n-2s)! (n-s)! s!} t^{n-2s},$$

( $t \in [-1, 1]$ )

The function  $P_n$ ,  $n=0, 1, \dots$ , given by

$$P_n(t) = P_{n0}(t) = \frac{1}{2^n} \sum_{s=0}^{[n/2]} (-1)^s \binom{n}{s} \binom{2n-2s}{n} t^{n-2s}.$$

( $t \in [-1, 1]$ )

is the well-known Legendre function.

$P_n$  is uniquely determined by the following properties:

(i)  $P_n$  is a polynomial of degree  $n$  in  $[-1,1]$ .

(ii)  $\int_{-1}^{+1} P_n(t) P_m(t) dt = 0$  for  $n \neq m$

(iii)  $P_n(1) = 1$ .

This is easily seen from the usual process of orthogonalization.

## 5. (Surface) Spherical Harmonics

Let  $S_n$  denote the space of all surface spherical harmonics of order  $n$ , i. e. the set of all restrictions

$$H_n|_{\Omega} \quad (5.1)$$

of the members  $H_n \in H_n$  to the unit sphere  $\Omega$ . As  $H_n$  is assumed to be homogeneous, the restriction  $H_n|_{\Omega}$  of  $H_n$  to  $\Omega$  is given by

$$H_n(\xi) = H_n\left(\frac{x}{|x|}\right), \quad |x| \neq 0, \quad \xi \in \Omega. \quad (5.2)$$

The restriction map

$$H_n \rightarrow H_n|_{\Omega}$$

is an isomorphism of  $H_n$  onto  $S_n$ , i. e. (i) each element  $S_n \in S_n$  is associated to one and only one element  $H_n \in H_n$  (ii) if the elements  $S_n, T_n \in S_n$  correspond to the elements  $H_n, K_n \in H_n$ , then the linear combination  $a S_n + b T_n \in S_n$  corresponds to the element  $a H_n + b K_n \in H_n$  (iii) different elements of  $S_n$  correspond to different elements of  $H_n$ .

Let us consider  $S_n$  as subspace of the space  $L^2(\Omega)$  of square - integrable functions on  $\Omega$  (equipped with the inner product  $(\cdot, \cdot)_{L^2(\Omega)}$ ). Then there are defined, for elements  $H_n \in H_n, K_n \in H_n$ , the fol-

lowing two inner products

$$(H_n, K_n)_{L^2(\Omega)} = \frac{1}{4\pi} \int_{\Omega} H_n(\xi) K_n(\xi) d\omega(\xi)$$

and

(5.3)

$$(H_n, K_n)_{P_n} = [H_n(\nabla_x)] K_n(x).$$

We now prove a theorem which is of basic importance for our task of exact computation of spherical harmonics:

For  $H_n \in H_n$ ,  $K_m \in H_m$ ,

$$(H_m, K_n)_{L^2(\Omega)} = \delta_{nm} \frac{n! \cdot 2^n}{(2n+1)!} [H_m(\nabla_x)] K_n(x),$$

(5.4)

$\delta_{nm}$  is the Kronecker-symbol.

**Proof.** By virtue of Green's formula it follows that

$$K_n(x) = \frac{1}{4\pi} \int_{\Omega} \left\{ \frac{1}{|x-y|} \frac{\partial}{\partial n_y} K_n(y) - K_n(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|} \right\} d\omega(y) \quad (5.4)$$

for all  $x \in R^3$  with  $|x| < 1$ , where  $\frac{\partial}{\partial n}$  denotes the derivative in the direction of the outer normal.

Therefore we find

$$[H_m(\nabla_x)] K_n(x) \quad (5.5)$$

$$= \frac{1}{4\pi} \int_{\Omega} \left\{ [H_m(\nabla_x)] \frac{1}{|x-y|} \frac{\partial}{\partial n_y} K_n(y) - K_n(y) \frac{\partial}{\partial n_y} [H_m(\nabla_x)] \frac{1}{|x-y|} \right\} d\omega(y)$$

For  $x \neq y$  we have

$$[H_m(\nabla_x)] \frac{1}{|x-y|} = (-1)^m \frac{(2m)!}{m! 2^m} \frac{H_m(x-y)}{|x-y|^{2m+1}} \quad (5.6)$$

This is equivalent to

$$[H_m(\nabla_x)] \frac{1}{|x-y|} = \frac{(2m)!}{m! 2^m} \frac{H_m(y-x)}{|x-y|^{2m+1}} \quad (5.7)$$

Inserting (5.7) into (5.5) gives

$$[H_m(\nabla_x)] K_n(x) \quad (5.8)$$

$$= \frac{(2m)!}{2^m(m!)} \frac{1}{4\pi} \int_{\Omega} \left\{ \frac{H_m(y-x)}{|x-y|^{2m+1}} \frac{\partial}{\partial n_y} K_n(y) - K_n(y) \frac{\partial}{\partial n_y} \frac{H_m(y-x)}{|x-y|^{2m+1}} \right\} d\omega(y)$$

It is easy to see that for  $m \neq n$

$$[H_m(\nabla_x)] K_n(x) \Big|_{x=0} = 0 \quad (5.9)$$

and for  $m = n$

$$[H_m(\nabla_x)] K_n(x) \Big|_{x=0} = [H_m(\nabla_x)] K_n(x) = (H_m, K_n) p_n \quad (5.10)$$

Therefore we obtain

$$\frac{1}{4\pi} \int_{\Omega} \left\{ \frac{H_m(y)}{|y|^{2m+1}} \frac{\partial}{\partial n_y} K_n(y) - K_n(y) \frac{\partial}{\partial n_y} \frac{H_m(y)}{|y|^{2m+1}} \right\} d\omega(y) \quad (5.11)$$

$$= \begin{cases} 0 & \text{for } m \neq n \\ \frac{2^m m!}{(2m)!} (H_m, K_n) p_n & \text{for } m = n \end{cases}.$$

Since the normal derivative of  $K_n(x)$  and  $H_m(x)$  are equal to

$$\frac{\partial}{\partial r} K_n(r\xi) \Big|_{r=1} = n K_n(\xi)$$

and (5.12)

$$\frac{\partial}{\partial r} H_m(r\xi) \Big|_{r=1} = m H_m(\xi)$$

respectively, it follows that

$$\frac{1}{4\pi} \int_{\Omega} \left\{ \frac{H_m(y)}{|y|^{2m+1}} \frac{\partial}{\partial n_y} K_n(y) - K_n(y) \frac{\partial}{\partial n_y} \frac{H_m(y)}{|y|^{2m+1}} \right\} d\omega(y) \quad (5.13)$$

$$= \frac{1}{4\pi} \int_{\Omega} \{ n H_m(\xi) K_n(\xi) + (m+1) H_m(\xi) K_n(\xi) \} d\omega(\xi)$$

$$= \frac{n+m+1}{4\pi} \int_{\Omega} H_m(\xi) K_n(\xi) d\omega(\xi).$$

Thus, by combination of (5.11) and (5.13), we finally obtain the desired result.

For  $H_n \in H_n$  ,  $K_m \in H_m$

$$(H_m, K_n)_{L^2(\Omega)} = \frac{\delta_{nm}}{\alpha_n} [H_n(\nabla_x)] K_n(x) .$$

$\alpha_n$  is given by

$$\alpha_n = \frac{(2n+1)!}{2^n n!} = 1 \cdot 3 \cdot \dots \cdot (2n+1) .$$

To any orthonormal system in  $H_n$  with respect to  $(\cdot, \cdot)_{P_n}$

$$\{K_{n,j}\}_{j=1, \dots, 2n+1} \quad (5.14)$$

there corresponds an orthonormal system in  $L^2(\Omega)$  with respect to  $(\cdot, \cdot)_{L^2(\Omega)}$

$$\{S_{n,j}^*\}_{j=1, \dots, 2n+1}$$

given by

$$S_{n,j}^* = \sqrt{\alpha_n} K_{n,j}|_{\Omega} , \quad (5.15)$$

and vice versa.

Hence, the addition theorem allows the following transcription into the space  $S_n$  of surface spherical harmonics:

Let  $\{S_{nj}^*\}_{j=1, \dots, 2n+1}$  be an orthonormal system in  $S_n$  with respect to  $(\cdot, \cdot)_{L^2(\Omega)}$ . Then, for  $\xi, \eta \in \Omega$ ,

$$\sum_{j=1}^{2n+1} S_{n,j}^*(\xi) S_{n,j}^*(\eta) = (2n+1) P_n(\xi\eta).$$

We discuss the most frequently used special case of the addition theorem.

Let

$$\begin{aligned} \xi &= (\sqrt{1-t^2} \cos \lambda, \sqrt{1-t^2} \sin \lambda, t)^T \\ \eta &= (\sqrt{1-s^2} \cos \phi, \sqrt{1-s^2} \sin \phi, s)^T. \end{aligned}$$

Setting  $t = \cos \theta$ ,  $s = \cos \tau$ ,  $0 \leq \theta, \tau \leq \pi$ , we obtain

$$\begin{aligned} \xi &= (\sin \theta \cos \lambda, \sin \theta \sin \lambda, \cos \theta)^T \\ \eta &= (\sin \tau \cos \phi, \sin \tau \sin \phi, \cos \tau)^T. \end{aligned}$$

( $0 \leq \lambda, \phi < 2\pi$ )

Thus it follows that

$$\begin{aligned} \xi \cdot \eta &= \sqrt{1-t^2} \sqrt{1-s^2} \cos(\lambda - \phi) + ts \\ &= \sin \theta \sin \tau \cos(\lambda - \phi) + \cos \theta \cos \tau. \end{aligned}$$



For our orthonormal system in  $S_n$  with respect to  $(\cdot, \cdot)_{L^2(\Omega)}$  we pick  $\{Y_{n,j}^*\}_{j=1, \dots, 2n+1}$  (cf. (0.5))

$$\begin{aligned} \sqrt{2n+1} P_{n,0}(t) &= P_{n,0}^*(t) \\ \sqrt{2(2n+1)} \frac{(n-1)!}{(n+1)!} P_{n1}(t) \cos \lambda &= P_{n,1}^*(t) \cos \lambda \\ \sqrt{2(2n+1)} \frac{(n-1)!}{(n+1)!} P_{n1}(t) \sin \lambda &= P_{n,1}^*(t) \sin \lambda \\ &\vdots \\ \sqrt{2(2n+1)} \frac{(n-n)!}{(n+n)!} P_{nn}(t) \cos(n\lambda) &= P_{n,n}^*(t) \cos(n\lambda) \\ \sqrt{2(2n+1)} \frac{(n-n)!}{(n+n)!} P_{nn}(t) \sin(n\lambda) &= P_{n,n}^*(t) \sin(n\lambda). \end{aligned}$$

Then, by virtue of the addition theorem, we obtain

$$\begin{aligned} &\sum_{j=1}^{2n+1} Y_{n,j}^*(\xi) Y_{n,j}^*(\eta) \\ &= (2n+1) P_n(\cos \theta) P_n(\cos \tau) \\ &+ 2(2n+1) \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_{nm}(\cos \theta) P_{nm}(\cos \tau) \cos m(\lambda - \phi). \end{aligned}$$

On the other hand, we have

$$P_n(\xi\eta) = P_n(\sin \theta \sin \tau \cos(\lambda - \phi) + \cos \theta \cos \tau).$$

Thus it follows that

$$P_n(\sin\theta \sin\tau \cos(\lambda-\phi) + \cos\theta \cos\tau) \quad (5.16)$$

$$= P_n(\cos\theta) P_n(\cos\tau) + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_{nm}(\cos\theta) P_{nm}(\cos\tau) \cos m(\lambda-\phi)$$

which is the classical addition theorem (decomposition formula) for Legendre functions.

Conventionally the functions  $P_{nm}^*$  defined by

$$P_{n0}^*(t) = \sqrt{2n+1} P_{n0}(t) \quad (5.17)$$

$$P_{nm}^*(t) = \sqrt{2(2n+1) \frac{(n-m)!}{(n+m)!}} P_{nm}(t)$$

and the trigonometric expressions

$$\begin{aligned} \sin(m\lambda) \\ \cos(m\lambda) \end{aligned} \quad (5.18)$$

are computed by recursion.

The generation of  $\sin(m\lambda)$  and  $\cos(m\lambda)$ ,  $m \geq 2$ , can be done through the following recursion relationships:

$$\begin{aligned} \sin(m\lambda) &= 2 \cos\lambda \sin((m-1)\lambda) - \sin((m-2)\lambda) \\ \cos(m\lambda) &= 2 \cos\lambda \cos((m-1)\lambda) - \cos((m-2)\lambda). \end{aligned}$$

These relationships are useful for point calculations, but they are inefficient for use if a set of points at a uniform longitude interval is being considered.

The generation of the functions  $P_{nm}^*$  is "critical to any calculation involving spherical harmonic expansions. In choosing an algorithm one must consider the speed and stability and accuracy of the procedure." (R. H. Rapp (1982)). The essential point is the sensitivity to small computational errors when  $n$  increases. In the past few years a number of different ways have been described in the literature. A comparison of methods has been given by C. C. Tscherning, R. H. Rapp, C. Goad (1983).

## 6. Exact Computation of Linearly Independent Systems of Homogeneous Harmonic Polynomials

Our purpose now is to explain the theoretical background of an alternate method for computation of spherical harmonics.

The concept is based on the observation that any linearly independent system

$$\{H_{n,j}\}_{j=1,\dots,2n+1} \quad (6.1)$$

of homogeneous harmonic polynomials of degree  $n$

$$\begin{array}{ccc} H_{n,1}(x) & = & \sum_{[\alpha]=n} C_{\alpha}^1 x^{\alpha} \\ \vdots & & \vdots \\ \vdots & & \vdots \end{array} \quad (6.2)$$

$$H_{n,2n+1}(x) = \sum_{[\alpha]=n} C_{\alpha}^{2n+1} x^{\alpha}$$

can be generated by exact computation of the coefficients  $C_{\alpha}^j$ ,  $j = 1, \dots, 2n+1$  (i.e. exclusively by integer operations).<sup>1)</sup> That is, the functions  $H_{n,j}$  are available globally in closed form as linear combination of monomials.

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1) We write briefly  $C_{\alpha}^j$  instead of  $C_{n;\alpha}^j$  when confusion is not likely to arise.

Given a homogeneous polynomial  $H_n$  of the form

$$H_n(x) = \sum_{[\alpha]=n} C_\alpha x^\alpha, \quad n \geq 2 \quad (6.3)$$

Assumed  $H_n$  is harmonic, i. e.,

$$\Delta_x H_n(x) = 0, \quad x \in \mathbb{R}^3, \quad (6.4)$$

we obtain

$$\begin{aligned} \Delta_x H_n(x) &= \Delta_x \sum_{[\alpha]=n} C_\alpha x^\alpha \\ &= \sum_{[\alpha]=n} C_\alpha \Delta_x (x^\alpha) = 0. \end{aligned} \quad (6.5)$$

Thus it follows that

$$\begin{aligned} \sum_{[\alpha]=n} C_\alpha [\alpha_1(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2}x_3^{\alpha_3} + \alpha_2(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2}x_3^{\alpha_3} + \alpha_3(\alpha_3-1)x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-2}] \\ = 0. \end{aligned} \quad (6.6)$$

We discuss the terms

$$\begin{aligned} \alpha_1 (\alpha_1-1) x_1^{\alpha_1-2} x_2^{\alpha_2} x_3^{\alpha_3} \\ \alpha_2 (\alpha_2-1) x_1^{\alpha_1} x_2^{\alpha_2-2} x_3^{\alpha_3} \\ \alpha_3 (\alpha_3-1) x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3-2} \end{aligned} \quad (6.7)$$

$$(\alpha_1 + \alpha_2 + \alpha_3 = n)$$

in more detail. Every term in (6.7) with index  $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$  satisfying  $\alpha_1 + \alpha_2 + \alpha_3 = n$  is a homogeneous polynomial of degree  $n-2$ . Hence, the sum

$$\Delta_x H_n(x) \quad (6.8)$$

$$= \sum_{[\alpha]=n} c_\alpha [\alpha_1(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2}x_3^{\alpha_3} + \alpha_2(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2}x_3^{\alpha_3} + \alpha_3(\alpha_3-1)x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-2}]$$

is a homogeneous polynomial of degree  $n-2$ . Therefore  $\Delta H_n$  can be represented in the form

$$\Delta_x H_n(x) = \sum_{[\beta]=n-2} D_\beta x^\beta. \quad (6.9)$$

The coefficients  $D_\beta$  are given by

$$D_\beta = \sum_{[\alpha]=n} c_\alpha m_{\beta\alpha} \quad (6.10)$$

where  $m_{\beta\alpha}$  is defined as follows

$$m_{\beta\alpha} = \begin{cases} \alpha_1(\alpha_1-1) & \text{for } \beta-\alpha = (-2, 0, 0)^T \\ \alpha_2(\alpha_2-1) & \text{for } \beta-\alpha = (0, -2, 0)^T \\ \alpha_3(\alpha_3-1) & \text{for } \beta-\alpha = (0, 0, -2)^T \\ 0 & \text{otherwise} \end{cases} \quad (6.11)$$

$H_n$  is assumed to be harmonic, i. e.  $\Delta_x H_n(x) = 0$  identically for all  $x \in \mathbb{R}^3$ . But this means all numbers  $D_\beta$  are equal to 0.

Hence, we have

$$\sum_{[\alpha]=n} C_\alpha m_{\beta\alpha} = 0 \quad (6.12)$$

for all  $\beta$  with  $[\beta] = n-2$ .

Now, (6.12) is a linear system of  $\binom{n}{2}$  equations in  $\binom{n+2}{2}$  unknowns  $C_\alpha$ ,  $[\alpha] = n$ .

The matrix

$$m = (m_{\beta\alpha}) \quad (6.13)$$

has  $\binom{n}{2}$  rows and  $\binom{n+2}{2}$  columns,  $m$  can be partitioned as follows

$$m = \left( \underbrace{1}_{\binom{n}{2}} : \underbrace{r}_{\binom{n+2}{2} - \binom{n}{2}} \right) \quad \left. \vphantom{\begin{matrix} 1 \\ r \end{matrix}} \right\} \binom{n}{2} \quad (6.14)$$

$$= 2n+1$$

where  $1 = (1_{\beta\delta})$  is a  $\binom{n}{2}$  by  $\binom{n}{2}$  matrix

and  $r = (r_{\beta\delta})$  is a  $\binom{n}{2}$  by  $\binom{n+2}{2} - \binom{n}{2}$

matrix.

For the set of multiindices of degree  $n$  we in-

introduce a binary relation between elements

$$\alpha' = (\alpha'_1, \alpha'_2, \alpha'_3)^T$$

$$\alpha'' = (\alpha''_1, \alpha''_2, \alpha''_3)^T$$

designated by " $>$ " and defined as follows:

$$\alpha' > \alpha'' \quad (6.15)$$

if and only if one of the following relations is satisfied

$$\alpha'_1 > \alpha''_1$$

or

$$\alpha'_1 = \alpha''_1, \alpha'_2 > \alpha''_2$$

or

$$\alpha'_1 = \alpha''_1, \alpha'_2 = \alpha''_2, \alpha'_3 > \alpha''_3.$$

The binary relation " $>$ " implies an ordering for the multiindices  $\alpha$ ,  $[\alpha] = n$  according to the mapping



$$\begin{array}{lll}
 (n, 0, 0) & \rightarrow & 1 \quad \} \quad 1 \\
 (n-1, 1, 0) & \rightarrow & 2 \quad \} \\
 (n-1, 0, 1) & \rightarrow & 3 \quad \} \quad 2 \\
 (n-2, 2, 0) & \rightarrow & 4 \quad \} \\
 (n-2, 1, 1) & \rightarrow & 5 \quad \} \quad 3 \\
 (n-2, 0, 2) & \rightarrow & 6 \quad \} \\
 \vdots & & \vdots \\
 (0, n, 0) & \rightarrow & \left( \begin{smallmatrix} n+2 \\ 2 \end{smallmatrix} \right) - n \quad \} \\
 \vdots & & \vdots \\
 (0, 0, n) & \rightarrow & \left( \begin{smallmatrix} n+2 \\ 2 \end{smallmatrix} \right) \quad \} \quad n+1
 \end{array}$$

In the same way, the set of multiindices  $\beta$ ,  $[\beta] = n-2$ , may be ordered by increasing integers  $i$ ,  $1 \leq i \leq \binom{n}{2}$ . Hence, in canonical manner, each pair  $(\beta, \alpha)$  with  $[\beta] = n-2$ ,  $[\alpha] = n$  corresponds uniquely to a pair  $(i, j)$ ,  $1 \leq i \leq \binom{n}{2}$ ,  $1 \leq j \leq \binom{n+2}{2}$ .

In this notation the matrix

$$m = (m_{\beta\alpha}) \quad , \quad [\beta] = n-2, \quad [\alpha] = n$$

can be rewritten in the ordered form

$$m = (m_{ij}) \quad , \quad 1 \leq i \leq \binom{n}{2} \quad , \quad 1 \leq j \leq \binom{n+2}{2}.$$

Analogously

$$l = (l_{\beta\gamma}) \quad , \quad [\beta] = n-2 \quad , \quad [\gamma] = n-2$$

becomes

$$l = (l_{ij}) \quad , \quad 1 \leq i \leq \binom{n}{2} \quad , \quad 1 \leq j \leq \binom{n}{2}.$$

From (6.11) it can be deduced that

$$l_{ij} = 0 \quad \text{for } i > j \quad , \quad i = 2, \dots, \binom{n}{2}$$

$$l_{ij} \neq 0 \quad \text{for } i = j \quad , \quad i = 1, \dots, \binom{n}{2}.$$

But this shows that  $l$  is non-singular, hence, the matrix  $m$  is of maximal rank:

$$\text{rk } (m) = \binom{n}{2}. \quad (6.16)$$

Therefore we are able to find  $\binom{n+2}{2} - \binom{n}{2}$ , i. e.  $2n+1$  linearly independent solution vectors

$$(A_{\alpha}^1) \quad , \quad \dots \quad , \quad (A_{\alpha}^{2n+1}) \quad , \quad [\alpha] = n,$$

of the homogeneous linear system (6.12).

According to standard conclusions in Linear Algebra the  $\binom{n+2}{2}$  by  $2n+1$  matrix  $a$  consisting of the vectors  $(A_{\alpha}^1) \quad , \quad \dots \quad , \quad (A_{\alpha}^{2n+1})$

$$a = \underbrace{((A_{\alpha}^1), \dots, (A_{\alpha}^{2n+1}))}_{2n+1} \left( \binom{n+2}{2} \right) = M \quad (6.17)$$

may be partitioned in the following form

$$a = \begin{bmatrix} u \\ -i \end{bmatrix} \begin{matrix} (n+2) \\ 2 \end{matrix} - (2n+1) \quad (6.18)$$

where  $i$  is the  $(2n+1)$  by  $(2n+1)$  unit matrix.

Then the linear system

$$ma = 0 \quad (6.19)$$

can be written as follows

$$lu = r. \quad (6.20)$$

Since  $l$  is a  $(2n+1)$  by  $(2n+1)$  upper triangular matrix, the unknown matrix  $u$  can be computed by  $(2n+1)$ -times backward substitution.

The elements of the matrix  $m$  are exclusively integers. Therefore, any solution  $A_\alpha$ ,  $[\alpha] = n$  of the linear system (6.19) is a column vector of rational components. Hence, there exists a matrix

$$c = ((C_\alpha^1), \dots, (C_\alpha^{2nn})) , [\alpha] = n,$$

the elements of which are exclusively integers (observe that if  $(A_\alpha)$ ,  $[\alpha] = n$ , is a solution of (6.19), then  $(C_\alpha) = k (A_\alpha)$ ,  $[\alpha] = n$ ,  $k$  integer, is a solution, too).

In other words, the solution process can be performed strictly in the modulus of integers.

Exact computation (without rounding errors) is possible in integer mode by use of integer operations (addition, subtraction, multiplication of integers).

When the matrix  $c$  has been calculated, the homogeneous harmonic polynomials  $H_{n,j}$  given by

$$H_{n,j}(x) = \sum_{[\alpha]=n} C_{\alpha}^j x^{\alpha}, \quad j = 1, \dots, 2n+1$$

form a (maximal) linearly independent system in  $H_n$ .

Using spherical coordinates

$$x = |x| (\sin\theta\cos\lambda, \sin\theta\sin\lambda, \cos\theta)^T$$

we obtain

$$H_{n,j}(x) = |x|^n \sum_{[\alpha]=n} C_{\alpha}^j (\sin^{\alpha_1+\alpha_2}\theta \cos^{\alpha_1}\lambda \sin^{\alpha_2}\lambda \cos^{\alpha_3}\theta)$$

The functions

$$\{H_{n,j}\}_{j=1, \dots, 2n+1}$$

given by

$$H_{n,j}(\xi) = \sum_{\alpha_1+\alpha_2+\alpha_3=n} C_{\alpha_1\alpha_2\alpha_3}^j \sin^{\alpha_1+\alpha_2}\theta \cos^{\alpha_1}\lambda \sin^{\alpha_2}\lambda \cos^{\alpha_3}\theta$$

$$(\xi \in \Omega, \xi = (\sin\theta\cos\lambda, \sin\theta\sin\lambda, \cos\theta)^T)$$

form a linearly independent system of surface spherical harmonics of order  $n$ .

**Computational Aspects:** In each row the matrix  $m$  contains three elements different from zero: the first of them is a diagonal element. Therefore, the matrix  $m$  (incl. its associated "index matrix") requires only

$$\frac{5}{2} n (n-1)$$

locations of storage if an optimal storage scheme is used. (Remark that complete storage needs

$$n^4 - 2n^3 - n^2 - 2n$$

locations).

If we look at the special structure of the solution matrix

$$c = ((C_{\alpha}^1), \dots, (C_{\alpha}^{2n+1})),$$

we have at most  $n^2 - n + 1$  non-vanishing elements in each column vector  $(C_{\alpha}^j)$ . Indeed, the total number is less. This gives rise for a further reduction in storage requirements.

Finally, it should be emphasized that exact computation, i. e. addition, subtraction, multiplication in integer mode must be performed strictly in the available range of the integer constants (determined by the architecture of the computer).

That means all data generated in the course of the calculations should be as small as possible. An efficient tool is decomposition into a product of prime numbers.

**Remark:** The loss of precision using real computation has not been investigated by the author.

**Examples:** Let us demonstrate the technique of calculating the matrix  $c$  for two examples.

**Example 1:**  $n = 3$

Then an elementary calculation yields

$$\begin{aligned} \binom{n+2}{2} &= 10 \\ \binom{n}{2} &= 3 \\ \binom{n+2}{2} - \binom{n}{2} &= 7 \end{aligned} .$$

Every homogeneous harmonic polynomial  $H_3 \in H_3$  may be represented in the form:

$$\begin{aligned} H_3(x) = & c_{300} x_1^3 + c_{210} x_1^2 x_2 + c_{201} x_1^2 x_3 \\ & + c_{120} x_1 x_2^2 + c_{111} x_1 x_2 x_3 + c_{102} x_1 x_3^2 \\ & + c_{030} x_2^3 + c_{021} x_2^2 x_3 + c_{012} x_2 x_3^2 \\ & + c_{003} x_3^3 . \end{aligned}$$

$$(x = (x_1, x_2, x_3)^T) .$$

$H_3$  has to fulfill the differential equation

$$\Delta_x H_3(x) = 0 ,$$

i. e.

$$\begin{aligned} & 6 C_{300} x_1 + 2 C_{210} x_2 + 2 C_{201} x_3 \\ & + 2 C_{120} x_1 + 6 C_{030} x_2 + 2 C_{021} x_3 \\ & + 2 C_{102} x_1 + 2 C_{012} x_2 + 6 C_{003} x_3 \\ & = 0 . \end{aligned}$$

Since  $\Delta_x H_3(x) = 0$  identically for all  $x \in R^3$  we get  
 $\binom{n}{2} = 3$  equations

$$\begin{aligned} 6 C_{300} + 2 C_{120} + 2 C_{102} &= 0 \\ 2 C_{210} + 6 C_{030} + 2 C_{012} &= 0 \\ 2 C_{201} + 2 C_{021} + 6 C_{003} &= 0 . \end{aligned}$$

Using the introduced order for the coefficients  $C_\alpha$ ,  
 $[\alpha] = 3$ , the equation  $mc = 0$  reads in matrix notation

$$\begin{bmatrix} 6 & 0 & 0 & : & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & : & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & : & 0 & 0 & 0 & 0 & 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} C_{300} \\ C_{210} \\ C_{201} \\ \dots \\ C_{120} \\ C_{111} \\ C_{102} \\ C_{030} \\ C_{021} \\ C_{012} \\ C_{003} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where we have marked the partitioning of the matrix  $m$  and the vector  $(C_\alpha)$  by dashed lines.

If we choose

$$C_{120} = -1, C_{111} = \dots = C_{003} = 0$$

the linear system is uniquely solved by the vector

$$\left(\frac{1}{3}, 0, 0 : -1, 0, 0, 0, 0, 0, 0\right)^T$$

Multiplying this vector by 3 all components become integers

$$(C_\alpha^1) = (1, 0, 0 : -3, 0, 0, 0, 0, 0, 0)^T$$

In the same way we generate a set of 7 linearly independent solutions of the above system the components of which are all integers, viz.



$$(c_{\alpha}^2) = (0, 0, 0, \vdots 0, -1, 0, 0, 0, 0, 0)^T$$

$$(c_{\alpha}^3) = (1, 0, 0, \vdots 0, 0, -3, 0, 0, 0, 0)^T$$

$$(c_{\alpha}^4) = (0, 3, 0, \vdots 0, 0, 0, -1, 0, 0, 0)^T$$

$$(c_{\alpha}^5) = (0, 0, 1, \vdots 0, 0, 0, 0, -1, 0, 0)^T$$

$$(c_{\alpha}^6) = (0, 1, 0, \vdots 0, 0, 0, 0, 0, -1, 0)^T$$

$$(c_{\alpha}^7) = (0, 0, 3, \vdots 0, 0, 0, 0, 0, 0, -1)^T.$$

Thus the following linearly independent system  $\{H_{3,j}\}_{j=1,\dots,7}$  of homogeneous harmonic polynomials of degree 3 is generated by the following functions:

$$H_{3,1}(x) = 1 \cdot x_1^3 - 3 x_1 x_2^2$$

$$H_{3,2}(x) = -1 \cdot x_1 x_2 x_3$$

$$H_{3,3}(x) = 1 \cdot x_1^3 - 3 x_1 x_3^2$$

$$H_{3,4}(x) = 3 x_1^2 x_2 - 1 \cdot x_2^3$$

$$H_{3,5}(x) = 1 \cdot x_1^2 x_3 - 1 \cdot x_2^2$$

$$H_{3,6}(x) = 1 \cdot x_1^2 x_2 - 1 \cdot x_2 x_3^2$$

$$H_{3,7}(x) = 3 x_1^2 x_3 - 1 \cdot x_3^3.$$

**Table: n = 3**

$$H_{3,1}(x) = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}$$


---

$$H_{3,2}(x) = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & 1 & 1 & 1 \end{array}$$


---

$$H_{3,3}(x) = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & 0 & 0 \\ -3 & 1 & 0 & 2 \end{array}$$


---

$$H_{3,4}(x) = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ 3 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \end{array}$$


---

$$H_{3,5}(x) = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{array}$$


---

$$H_{3,6}(x) = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 2 \end{array}$$


---

$$H_{3,7}(x) = \begin{array}{cccc} & x_1 & x_2 & x_3 \\ 3 & 2 & 0 & 1 \\ -1 & 0 & 0 & 3 \end{array}$$


---

**Example 2: n = 5**

Now we have

$$\binom{n+2}{2} = 21$$

$$\binom{n}{2} = 10$$

Hence,

$$\binom{n+2}{2} - \binom{n}{2} = 11$$

Every homogeneous polynomial of degree 5 may be written in the form

$$\begin{aligned} H_5(x) &= \sum_{[\alpha]=5} C_\alpha x^\alpha \\ &= \sum_{\alpha_1+\alpha_2+\alpha_3=5} C_{\alpha_1\alpha_2\alpha_3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} . \end{aligned}$$

Explicitly,

$$\begin{aligned} H_5(x) &= C_{500} x_1^5 + C_{410} x_1^4 x_2 + C_{401} x_1^4 x_3 \\ &+ C_{320} x_1^3 x_2^2 + C_{311} x_1^3 x_2 x_3 + C_{302} x_1^3 x_3^2 \\ &+ C_{230} x_1^2 x_2^3 + C_{211} x_1^2 x_2^2 x_3 + C_{212} x_1^2 x_2 x_3^2 \\ &+ C_{203} x_1^2 x_3^3 + C_{140} x_1 x_2^4 + C_{131} x_1 x_2^3 x_3 \\ &+ C_{122} x_1 x_2^2 x_3^2 + C_{113} x_1 x_2 x_3^3 + C_{104} x_1 x_3^4 \\ &+ C_{050} x_2^5 + C_{041} x_2^4 x_3 + C_{032} x_2^3 x_3^2 \\ &+ C_{023} x_2^2 x_3^3 + C_{014} x_2 x_3^4 + C_{005} x_3^5 . \end{aligned}$$

Hence, the equation

$$\Delta_x H_5(x) = 0, \quad x \in \mathbb{R}^3$$

is equivalent to

$$\begin{aligned} & 20 c_{500} x_1^3 + 12 c_{410} x_1^2 x_2 + 12 c_{401} x_1^2 x_3 + 6 c_{320} x_1 x_2^2 + 6 c_{311} x_1 x_2 x_3 \\ & + 2 c_{320} x_1^3 + 6 c_{230} x_1^2 x_2 + 2 c_{221} x_1^2 x_3 + 12 c_{140} x_1 x_2^2 + 6 c_{131} x_1 x_2 x_3 \\ & + 2 c_{302} x_1^3 + 2 c_{210} x_1^2 x_2 + 6 c_{203} x_1^2 x_3 + 2 c_{122} x_1 x_2^2 + 6 c_{113} x_1 x_2 x_3 \\ & + 6 c_{302} x_1 x_2^2 + 2 c_{230} x_2^3 + 2 c_{211} x_2^2 x_3 + 2 c_{212} x_2 x_3^2 + 2 c_{203} x_3^3 \\ & + 2 c_{122} x_1 x_3^2 + 20 c_{050} x_2^3 + 12 c_{041} x_2^2 x_3 + 6 c_{032} x_2 x_3^2 + 2 c_{023} x_3^3 \\ & + 12 c_{104} x_1 x_3^2 + 2 c_{032} x_2^3 + 6 c_{023} x_2^2 x_3 + 12 c_{014} x_2 x_3^2 + 20 c_{005} x_3^3 \\ & = 0. \end{aligned}$$

But this gives

$$\begin{aligned} 20 c_{500} + 2 c_{320} + 2 c_{302} &= 0 \\ 12 c_{410} + 6 c_{230} + 2 c_{212} &= 0 \\ 12 c_{401} + 2 c_{221} + 6 c_{203} &= 0 \\ 6 c_{320} + 12 c_{140} + 2 c_{122} &= 0 \\ 6 c_{311} + 6 c_{131} + 6 c_{113} &= 0 \\ 6 c_{302} + 2 c_{122} + 12 c_{104} &= 0 \end{aligned}$$

$$2 C_{230} + 20 C_{050} + 2 C_{032} = 0$$

$$2 C_{211} + 12 C_{041} + 6 C_{023} = 0$$

$$2 C_{212} + 6 C_{032} + 12 C_{014} = 0$$

$$2 C_{203} + 2 C_{023} + 20 C_{005} = 0$$

These equations can be rewritten in matrix notation as follows

$$\begin{bmatrix} 20 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} C_{500} \\ C_{410} \\ C_{401} \\ C_{320} \\ C_{311} \\ C_{302} \\ C_{221} \\ C_{212} \\ C_{203} \\ C_{140} \\ C_{131} \\ C_{122} \\ C_{113} \\ C_{104} \\ C_{050} \\ C_{041} \\ C_{032} \\ C_{023} \\ C_{014} \\ C_{005} \end{bmatrix} = 0,$$

where we have again marked the partitioning of the matrix and the vector in the indicated way.

We choose  $C_{140} = -1$  ,  $C_{131} = \dots = C_{005} = 0$ . Then the linear system is uniquely solvable by the the vector

$$(-\frac{1}{5}, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, -1, 0, \dots, 0)^T$$

Multiplying this vector by the factor 5 all components become integers

$$(c^1_\alpha) = (-1, 0, 0, 10, 0, 0, 0, 0, 0, 0, 0, 0, -5, 0, \dots, 0)^T$$

In the same way we generate a set of 11 linearly independent solutions of the above system the components of which are integers by

- (i) choosing the lower part of the vector identically 0 besides one component
- (ii) solving the system by backward substitution
- (iii) multiplying every resulting vector by an appropriate integer.

According to this procedure the following system is generated

$$H_{5,1}(x) = \begin{array}{rcccc} & & x_1 & x_2 & x_3 \\ -1 & 5 & 0 & 0 \\ +10 & 3 & 2 & 0 \\ -5 & 1 & 4 & 0 \end{array}$$


---

$$H_{5,2}(x) = \begin{array}{rcccc} & & x_1 & x_2 & x_3 \\ +1 & 3 & 1 & 1 \\ -1 & 1 & 3 & 1 \end{array}$$


---

$$H_{5,3}(x) = \begin{array}{rcccc} & & x_1 & x_2 & x_3 \\ -1 & 5 & 0 & 0 \\ +5 & 3 & 2 & 0 \\ +5 & 3 & 0 & 2 \\ -15 & 1 & 2 & 2 \end{array}$$


---

$$H_{5,4}(x) = \begin{array}{rcccc} & & x_1 & x_2 & x_3 \\ +1 & 3 & 1 & 1 \\ -1 & 1 & 1 & 3 \end{array}$$


---

$$H_{5,5}(x) = \begin{array}{rcccc} & & x_1 & x_2 & x_3 \\ -1 & 5 & 0 & 0 \\ +10 & 3 & 0 & 2 \\ -5 & 1 & 0 & 4 \end{array}$$


---

$$H_{5,6}(x) = \begin{array}{rcccc} & & x_1 & x_2 & x_3 \\ -5 & 4 & 1 & 0 \\ +10 & 2 & 3 & 0 \\ -1 & 0 & 5 & 0 \end{array}$$


---

$$H_{5,7}(x) = \begin{array}{rcccc} & & x_1 & x_2 & x_3 \\ -1 & 4 & 0 & 1 \\ +6 & 2 & 2 & 1 \\ -1 & 0 & 4 & 1 \end{array}$$


---

$$\begin{array}{rcccc}
 & & x_1 & x_2 & x_3 \\
 H_{5,8}(x) = & -1 & 4 & 1 & 0 \\
 & +1 & 2 & 3 & 0 \\
 & +3 & 2 & 1 & 2 \\
 & -1 & 0 & 3 & 2
 \end{array}$$


---

$$\begin{array}{rcccc}
 & & x_1 & x_2 & x_3 \\
 H_{5,9}(x) = & -1 & 4 & 0 & 1 \\
 & +3 & 2 & 2 & 1 \\
 & +1 & 2 & 0 & 3 \\
 & -1 & 0 & 2 & 3
 \end{array}$$


---

$$\begin{array}{rcccc}
 & & x_1 & x_2 & x_3 \\
 H_{5,10}(x) = & -1 & 4 & 1 & 0 \\
 & +6 & 2 & 1 & 2 \\
 & -1 & 0 & 1 & 4
 \end{array}$$


---

$$\begin{array}{rcccc}
 & & x_1 & x_2 & x_3 \\
 H_{5,11}(x) = & -5 & 4 & 0 & 1 \\
 & +10 & 2 & 0 & 3 \\
 & -1 & 0 & 0 & 5
 \end{array}$$


---

In Appendix 1 we give a list for the first linearly independent systems of homogeneous harmonic polynomials  $H_{n,j}$  for  $n = 3$  through  $n = 10$ .



## 7. Exact Computation of Orthonormal Systems of Homogeneous Harmonic Polynomials

Corresponding to the linearly independent system of homogeneous harmonic polynomials of degree  $n$

$$\{H_{n,j}\}_{j=1,\dots,2n+1}$$

an orthogonal system

$$\{\tilde{H}_{n,j}\}_{j=1,\dots,2n+1}$$

can be constructed as usual (according to the well-known Gram-Schmidt process):

The functions  $\tilde{H}_{nj}$  are computed recursively. We start from

$$\tilde{H}_{n,1} = H_{n,1} \quad . \quad (7.1)$$

Then we set

$$\tilde{H}_{n,2} = a_{2,1}^n \tilde{H}_{n,1} + H_{n,1} \quad . \quad (7.2)$$

The coefficient  $a_{2,1}^n$  has to be chosen such that  $\tilde{H}_{n,2}$  is orthogonal to  $\tilde{H}_{n1}$ :

$$(\tilde{H}_{n,2}, \tilde{H}_{n,1})_{P_n} = 0 \quad . \quad (7.3)$$

It turns out that

$$a_{2,1}^n = - \frac{(H_{n,2}, \tilde{H}_{n,1}) p_n}{(\tilde{H}_{n,1}, \tilde{H}_{n,1}) p_n} . \quad (7.4)$$

It should be noted that numerator and denominator may be determined exactly (cf. (2.5)).

Now, let

$$\tilde{H}_{n,3} = a_{3,1}^n \tilde{H}_{n,1} + a_{3,2}^n \tilde{H}_{n,2} + H_{n,3} . \quad (7.5)$$

The requirements

$$(\tilde{H}_{n,3}, \tilde{H}_{n,1}) p_n = 0 \quad (7.6)$$

$$(\tilde{H}_{n,3}, \tilde{H}_{n,2}) p_n = 0$$

lead to

$$a_{3,1}^n = - \frac{(H_{n,3}, \tilde{H}_{n,1}) p_n}{(\tilde{H}_{n,1}, \tilde{H}_{n,1}) p_n} \quad (7.7)$$

$$a_{3,2}^n = - \frac{(H_{n,3}, \tilde{H}_{n,2}) p_n}{(\tilde{H}_{n,2}, \tilde{H}_{n,2}) p_n} .$$

Again, the coefficients can be deduced exclusively by integer operations.

Analogously we get, in general,

$$\tilde{H}_{n,k} = a_{k,1}^n \tilde{H}_{n,1} + \dots + a_{k,k-1}^n \tilde{H}_{n,k-1} + H_{n,k} \quad (7.8)$$

$$(k = 2, \dots, 2n+1)$$

$$\tilde{H}_{n,1} = H_{n,1} \quad , \quad (7.9)$$

where the coefficients

$$a_{k,s}^n = - \frac{(H_{n,k} , \tilde{H}_{n,s}) p_n}{(\tilde{H}_{n,s} , \tilde{H}_{n,s}) p_n} \quad (7.10)$$

are computable exactly by integer operations, i. e.  $a_{k,s}^n$  is known exactly as fraction of integers.

According to the orthogonalization scheme, each function  $\tilde{H}_{n,j}$  is a linear combination of the functions

$$H_{n,1} , \dots , H_{n,2n+1} .$$

The coefficients of this linear combination can be obtained exactly as rational numbers, too. Thus there exists a vector

$$(\tilde{C}_\alpha^j)$$

such that

$$\tilde{H}_{n,j}(x) = \sum_{[\alpha]=n} \tilde{C}_\alpha^j x^\alpha \quad , \quad j = 1, \dots, 2n+1 \quad . \quad (7.11)$$

The vectors  $(\tilde{C}_\alpha^j)$ ,  $j=1, \dots, 2n+1$ , form a matrix  $\tilde{c}$  exclusively consisting of fractions of integers (assumed all numbers in the course of computation have been calculated in such a way that numerator and denominator are known as integers).

There exists a sequence of homogeneous harmonic polynomials

$$\{\tilde{H}_{n,j}\}_{j=1, \dots, 2n+1}$$

with

$$(\tilde{H}_{n,j}, \tilde{H}_{n,1})_{P_n} = 0 \quad \text{for } j \neq 1,$$

viz.:

$$\tilde{H}_{n,k} = a_{k,1}^n \tilde{H}_{n,1} + \dots + a_{k,k-1}^n \tilde{H}_{n,k-1} + H_{n,k} \\ (k = 2, \dots, 2n+1)$$

$$\tilde{H}_{n,1} = H_{n,1}.$$

In addition, a sequence

$$\{K_{n,j}\}_{j=1, \dots, 2n-1} \tag{7.12}$$

can be constructed so as to have the property

$$(K_{n,j}, K_{n,1})_{P_n} = \delta_{j1} = \begin{cases} 1 & \text{for } j = 1 \\ 0 & \text{for } j \neq 1 \end{cases}.$$

The functions  $K_{n,j}$  can be obtained by dividing each element  $\tilde{H}_{n,j}$  by its norm

$$K_{n,j} = \frac{\tilde{H}_{n,j}}{\sqrt{(\tilde{H}_{n,j}, \tilde{H}_{n,j})_{P_n}}}.$$

The sequence

$$\{K_{n,j}\}_{j=1,\dots,2n+1}$$

forms an orthonormal system of homogeneous harmonic polynomials of degree  $n$ .

Provided the expression

$$\sqrt{(\tilde{H}_{n,j}, \tilde{H}_{n,j})_{P_n}}$$

has been stored as radicant of an integer, the functions

$$\{K_{n,j}\}_{j=1,\dots,2n+1}$$

are available exactly, too. The exactness is of basic importance. It implies the stability of the solution process. In fact, the method is constructed so as to have no sensitivity to computational errors.

To any orthonormal system of homogeneous harmonic polynomials  $\{K_{n,j}\}_{j=1,\dots,2n+1}$  in  $(H_n, (\cdot, \cdot)_{P_n})$

there corresponds an orthonormal system of spherical harmonics  $\{S_{n,j}^*\}_{j=1,\dots,2n+1}$  in  $(S_n, (\cdot, \cdot)_{L^2(\Omega)})$  given by

$$S_{n,j}^* = \sqrt{\alpha_n} K_{n,j} | \Omega \quad . \quad (7.13)$$

According to our orthonormalization process, each element  $S_{n,j}^*$  is available as follows

$$S_{n,j}^*(\xi) = \sum_{[\alpha]=n} B_{\alpha}^j \xi^{\alpha} \quad , \quad \xi \in \Omega \quad , \quad (7.14)$$

where we have set

$$B_{\alpha}^j = \frac{\sqrt{\alpha_n}}{\sqrt{(\tilde{H}_{n,j}, \tilde{H}_{n,j})_{P_n}}} \tilde{C}_{\alpha}^j \quad . \quad (7.15)$$

Hence, the radical sign and the division are the only sources for rounding errors.

Since spherical harmonics of different order are orthogonal we finally find:

The system

$$S_{n,j}^*(\xi) = \frac{\sqrt{\alpha_n}}{\sqrt{(\tilde{H}_{n,j}, \tilde{H}_{n,j})_{P_n}}} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3} \tilde{C}_{\alpha_1 \alpha_2 \alpha_3}^j \sin^{(\alpha_1 + \alpha_2)} \theta \cos^{\alpha_1} \lambda \sin^{\alpha_2} \lambda \cos^{\alpha_3} \theta$$

$$(j = 1, \dots, 2n+1 ; \xi = (\sin \theta \cos \lambda, \sin \theta \sin \lambda, \cos \theta)^T)$$

is orthonormal in the sense

$$(S_{n,1}^*, S_{k,j}^*)_{L^2(\Omega)} = \delta_{nk} \cdot \delta_{1j}.$$

Summarizing our results we obtain:

Each element  $S_{n,j}^*$  of the orthonormal system

$$\{S_{n,j}^*\}_{j=1,\dots,2n+1}$$

of surface spherical harmonics of order  $n$  may be represented in the form

$$S_{n,j}^*(\xi) = \sum_{[\alpha]=n} b_{\alpha}^j \xi^{\alpha}, \quad \xi \in \Omega,$$

$$= \sum_{\alpha_1+\alpha_2+\alpha_3=n} b_{\alpha_1\alpha_2\alpha_3}^j \sin^{(\alpha_1+\alpha_2)} \theta \cos^{\alpha_1} \lambda \sin^{\alpha_2} \lambda \cos^{\alpha_3} \theta$$

$$(j = 1, \dots, 2n+1, \xi = (\sin\theta\cos\lambda, \sin\theta\sin\lambda, \cos\theta)^T),$$

where the coefficients  $B_{\alpha}^j$  are given as follows

$$B_{\alpha}^j = \sqrt{\frac{\alpha_n}{(\tilde{H}_{n,j}, \tilde{H}_{n,j})_{P_n}}} \tilde{C}_{\alpha}^j,$$

and the values

$$(\tilde{H}_{n,j}, \tilde{H}_{n,j})_{P_n} \text{ and } \tilde{C}_{\alpha}^j$$

can be determined exclusively by integer operations.

**Examples:** Again, we discuss the orders  $n = 3$  and  $n = 5$ .

**Example 1:**  $n = 3$

According to the aforementioned orthonormalization process due to Gram - Schmidt we are able to deduce from the maximal system of linearly independent homogeneous harmonic polynomials

$$\{H_{3,j}\}_{j=1,\dots,7}$$

an orthogonal system

$$\{\tilde{H}_{3,j}\}_{j=1,\dots,7}.$$

The resulting functions are listed below:



$$\begin{aligned}
 \bar{H}_{3,1}(x) &= x_1^3 - 3 x_1 x_2^2 \\
 \bar{H}_{3,2}(x) &= x_1 x_2 x_3 \\
 \bar{H}_{3,3}(x) &= x_1^3 + x_1 x_2^2 - 4 x_1 x_3^2 \\
 \bar{H}_{3,4}(x) &= 3 x_1^2 x_2 - x_2^3 \\
 \bar{H}_{3,5}(x) &= x_1^2 x_3 - x_2^2 x_3 \\
 \bar{H}_{3,6}(x) &= x_1^2 x_2 + x_2^3 - 4 x_2 x_3^2 \\
 \bar{H}_{3,7}(x) &= 3 x_1^2 x_3 + 3 x_2^2 x_3 - 2 x_3^3 .
 \end{aligned}$$

These functions may be rewritten in the following form:

$$\begin{aligned}
 \bar{H}_{3,1}(x) &= 1 \cdot 2^0 \cdot 3^0 \cdot x_1^3 x_2^0 x_3^0 \\
 &\quad + (-1) \cdot 2^0 \cdot 3^1 \cdot x_1^1 x_2^2 x_3^0 \\
 \hline
 \bar{H}_{3,2}(x) &= 1 \cdot 2^0 \cdot 3^0 \cdot x_1^1 x_2^1 x_3^1 \\
 \hline
 \bar{H}_{3,3}(x) &= 1 \cdot 2^0 \cdot 3^0 \cdot x_1^3 x_2^0 x_3^0 \\
 &\quad + 1 \cdot 2^0 \cdot 3^0 \cdot x_1^1 x_2^2 x_3^0 \\
 &\quad + (-1) \cdot 2^2 \cdot 3^0 \cdot x_1^1 x_2^0 x_3^2 \\
 \hline
 \bar{H}_{3,4}(x) &= 1 \cdot 2^0 \cdot 3^1 \cdot x_1^2 x_2^1 x_3^0 \\
 &\quad + (-1) \cdot 2^0 \cdot 3^0 \cdot x_1^0 x_2^3 x_3^0 \\
 \hline
 \end{aligned}$$

$$\begin{aligned}\tilde{H}_{3,5}(x) &= 1 \cdot 2^0 \cdot 3^0 \cdot x_1^2 x_2^0 x_3^1 \\ &+ (-1) \cdot 2^0 \cdot 3^0 \cdot x_1^0 x_2^2 x_3^1\end{aligned}$$


---

$$\begin{aligned}\tilde{H}_{3,6}(x) &= 1 \cdot 2^0 \cdot 3^0 \cdot x_1^2 x_2^1 x_3^0 \\ &+ 1 \cdot 2^0 \cdot 3^0 \cdot x_1^0 x_2^3 x_3^0 \\ &- (-1) \cdot 2^2 \cdot 3^0 \cdot x_1^0 x_2^1 x_3^2\end{aligned}$$


---

$$\begin{aligned}\tilde{H}_{3,7}(x) &= 1 \cdot 2^0 \cdot 3^1 \cdot x_1^2 x_2^0 x_3^1 \\ &+ 1 \cdot 2^0 \cdot 3^1 \cdot x_1^0 x_2^2 x_3^1 \\ &+ (-1) \cdot 2^1 \cdot 3^0 \cdot x_1^0 x_2^0 x_3^3\end{aligned}$$


---

That means, all components  $\tilde{C}_\alpha^j \neq 0$  are decomposed into an integer times a product of the prime numbers 2, 3.

**Table:**  $n = 3$

	2	3	$x_1$	$x_2$	$x_3$
$\tilde{H}_{3,1}(x)$	1	0	0	3	0
	-1	0	1	1	2
$\tilde{H}_{3,2}(x)$	1	0	0	1	1

		2	3	$x_1$	$x_2$	$x_3$
$\tilde{H}_{3,3}(x)$	=	1	0	0	3	0
		1	0	0	1	2
		-1	2	0	1	0
$\tilde{H}_{3,4}(x)$	=	1	0	1	2	1
		-1	0	0	0	3
$\tilde{H}_{3,5}(x)$	=	1	0	0	2	0
		-1	0	0	0	2
$\tilde{H}_{3,6}(x)$	=	1	0	0	2	1
		1	0	0	0	3
		-1	2	0	0	1
$\tilde{H}_{3,7}(x)$	=	1	0	1	2	0
		1	0	1	0	2
		-1	1	0	0	0

An easy calculation gives

$$\begin{aligned}
 (\tilde{H}_{3,1}, \tilde{H}_{3,1})_{P_3} &= 24 = 1 \cdot 2^3 \cdot 3^1 \\
 (\tilde{H}_{3,2}, \tilde{H}_{3,2})_{P_3} &= 1 = 1 \cdot 2^0 \cdot 3^0 \\
 (\tilde{H}_{3,3}, \tilde{H}_{3,3})_{P_3} &= 40 = 5 \cdot 2^3 \cdot 3^0 \\
 (\tilde{H}_{3,4}, \tilde{H}_{3,4})_{P_3} &= 24 = 1 \cdot 2^3 \cdot 3^1 \\
 (\tilde{H}_{3,5}, \tilde{H}_{3,5})_{P_3} &= 4 = 1 \cdot 2^2 \cdot 3^0 \\
 (\tilde{H}_{3,6}, \tilde{H}_{3,6})_{P_3} &= 40 = 5 \cdot 2^3 \cdot 3^0 \\
 (\tilde{H}_{3,7}, \tilde{H}_{3,7})_{P_3} &= 60 = 5 \cdot 2^2 \cdot 3^1
 \end{aligned}$$

Thus, the integers are decomposed into a (positive) integer times a product of prime numbers  $\leq 3$ .

Consequently, the orthonormal system

$$\{K_{3,j}\}_{j=1,\dots,7}$$

corresponding to

$$\{\tilde{H}_{n,j}\}_{j=1,\dots,7}$$

may be listed as follows:

Table:  $n = 3$

	2	3	$x_1$	$x_2$	$x_3$	
$K_{3,1}(x)$	=					
	1	0	0	3	0	0
	-1	0	1	1	2	0
	1	3	1	: normalization-factor		
$K_{3,2}(x)$	=					
	1	0	0	1	1	1
	1	0	0			
$K_{3,3}(x)$	=					
	1	0	0	3	0	0
	1	0	0	1	2	0
	-1	2	0	1	0	2
	5	3	0			

$$\begin{array}{rcccccc}
 & & 2 & 3 & x_1 & x_2 & x_3 \\
 \hline
 K_{3,4}(x) & = & & & & & \\
 & & 1 & 0 & 1 & 2 & 1 & 0 \\
 & & -1 & 0 & 0 & 0 & 3 & 0 \\
 & & 1 & 3 & 1 & & & 
 \end{array}$$

$$\begin{array}{rcccccc}
 K_{3,5}(x) & = & & & & & \\
 & & 1 & 0 & 0 & 2 & 0 & 1 \\
 & & -1 & 0 & 0 & 0 & 2 & 1 \\
 & & 1 & 2 & 0 & & & 
 \end{array}$$

$$\begin{array}{rcccccc}
 K_{3,6}(x) & = & & & & & \\
 & & 1 & 0 & 0 & 2 & 1 & 0 \\
 & & 1 & 0 & 0 & 0 & 3 & 0 \\
 & & -1 & 2 & 0 & 0 & 1 & 2 \\
 & & 5 & 3 & 0 & & & 
 \end{array}$$

$$\begin{array}{rcccccc}
 K_{3,7}(x) & = & & & & & \\
 & & 1 & 0 & 1 & 2 & 0 & 1 \\
 & & 1 & 0 & 1 & 0 & 2 & 1 \\
 & & -1 & 1 & 0 & 0 & 0 & 3 \\
 & & 5 & 2 & 1 & & & 
 \end{array}$$

For example,  $K_{3,7}(x)$  reads explicitly

$$\begin{aligned}
 K_{3,7}(x) = & ( 1 \cdot 2^0 \cdot 3^1 \cdot x_1^2 x_2^0 x_3^1 \\
 & + 1 \cdot 2^0 \cdot 3^1 \cdot x_1^0 x_2^2 x_3^1 \\
 & - 1 \cdot 2^1 \cdot 3^0 \cdot x_1^0 x_2^0 x_3^3 ) / \sqrt{5 \cdot 2^2 \cdot 3^1} .
 \end{aligned}$$

Finally, the orthonormal system

$$\{S_{3,j}^*\}_{j=1,\dots,7}$$

of surface spherical harmonics of degree  $n$  (with respect to  $(\cdot, \cdot)_{L^2(\Omega)}$ ) is given as follows

$$S_{3,j}^*(\xi) = \sqrt{\alpha_3} K_{3,j} \left( \frac{x}{|x|} \right), \quad (x = |x| \xi, \xi \in \Omega)$$

with

$$\alpha_3 = 105 = 1 \cdot 3 \cdot 5 \cdot 7.$$

**Example 2:**  $n = 5$

Analogous calculations give the following functions:

**Table:**  $n = 5$

	2	3	5	$x_1$	$x_2$	$x_3$	
$K_{5,1}(x) =$	-1	0	0	0	5	0	0
	1	1	0	1	3	2	0
	-1	0	0	1	1	4	0
	1	7	1	1	: normalization-factor		
$K_{5,2}(x) =$	1	0	0	0	3	1	1
	-1	0	0	0	1	3	1
	1	2	1	0			

	2	3	5	$x_1$	$x_2$	$x_3$
$K_{5,3}(x) =$	-1	0	0	0	5	0
	1	1	0	0	3	2
	1	3	0	0	3	0
	1	0	1	0	1	4
	-1	3	1	0	1	2
	1	7	3	0		

$K_{5,4}(x) =$	1	0	0	0	3	1
	1	0	0	0	1	3
	-1	1	0	0	1	1
	1	2	2	0		

$K_{5,5}(x) =$	-1	0	0	0	5	0
	-1	1	0	0	3	2
	1	2	1	0	3	0
	-1	0	0	0	1	4
	1	2	1	0	1	2
	-1	3	0	0	1	0
	7	6	2	0		

$K_{5,6}(x) =$	-1	0	0	1	4	1
	1	1	0	1	2	3
	-1	0	0	0	0	5
	1	7	1	1		

	2	3	5	$x_1$	$x_2$	$x_3$
$K_{5,7}(x) =$	-1	0	0	0	4	0
	1	1	1	0	2	2
	-1	0	0	0	0	4
	1	6	1	0		

$K_{5,8}(x) =$	-1	0	1	0	4	1	0
	-1	1	0	0	2	3	0
	1	3	1	0	2	1	2
	1	0	0	0	0	5	0
	-1	3	0	0	0	3	2
	1	7	3	0			

$K_{5,9}(x) =$	-1	0	0	0	4	0	1
	1	1	0	0	2	0	3
	1	0	0	0	0	4	1
	-1	1	0	0	0	2	3
	1	4	2	0			

$K_{5,10}(x) =$	-1	0	0	0	4	1	0
	-1	1	0	0	2	3	0
	1	2	1	0	2	1	2
	-1	0	0	0	0	5	0
	1	2	1	0	0	3	2
	-1	3	0	0	0	1	4
	7	6	2	0			



	2	3	5	$x_1$	$x_2$	$x_3$
$K_{5,11}(x) =$						
	-1	0	1	1	4	0
	-1	1	1	1	2	2
	1	3	0	1	2	0
	-1	0	1	1	0	4
	1	3	0	1	0	2
	-1	3	0	0	0	0
	7	6	3	1		

The functions  $\{S_{5,j}^*\}_{j=1,\dots,11}$  now are given as follows

$$S_{5,j}^*(\xi) = \sqrt{\alpha_5} K_{5,j} \left( \frac{x}{|x|} \right), \quad (x = |x| \xi, \quad \xi \in \Omega)$$

with

$$\alpha_5 = 10395 = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11.$$

A list of the first orthonormal systems of homogeneous harmonic polynomials is given in Appendix 2. In the same way as illustrated above, the coefficients  $\tilde{C}_\alpha^j$  and the normalization-factor are split into an integer times a product of prime numbers  $\leq n$ .

## 8. Relations to the Standard System of Spherical Harmonics

Let

$$\{H_{n,j}\}_{j=1,\dots,2n+1} \quad (n \geq 2) \quad (8.1)$$

be the sequence of linearly independent homogeneous harmonic polynomials of degree  $n$  generated by exact computation as described in chapter 6:

$$H_{n,j}(x) = \sum_{[\alpha]=n} C_{\alpha}^j x^{\alpha}, \quad x \in \mathbb{R}^3, \quad (8.2)$$

where the column vectors  $(C_{\alpha}^j)$  of the matrix

$$c = ((C_{\alpha}^1), \dots, (C_{\alpha}^{2n+1})) \quad (8.3)$$

consist exclusively of integers. (Remember that the matrix  $c$  has the form

$$c = \begin{pmatrix} p \\ q \end{pmatrix}_{2n+1} \begin{matrix} M-(2n+1) \\ 2n+1 \end{matrix}, \quad (8.4)$$

where  $M$  is equal to  $\binom{n+2}{2}$  and  $q$  is a  $(2n+1)$  by  $(2n+1)$  diagonal matrix with integers different from zero on its diagonal).

It is clear that the system

$$\{H_{n,j}|\Omega\}_{j=1,\dots,2n+1} \quad (8.5)$$

given by

$$\begin{aligned} H_{n,1}(\xi) &= \sum_{[\alpha]=n} C_{\alpha}^1 \xi^{\alpha}, \quad \xi \in \Omega \\ &\vdots \\ H_{n,2n+1}(\xi) &= \sum_{[\alpha]=n} C_{\alpha}^{2n+1} \xi^{\alpha}, \quad \xi \in \Omega \end{aligned} \quad (8.6)$$

forms a maximal linearly independent set of (surface) spherical harmonics of order  $n$ .

On the other hand we know that the standard system

$$\begin{aligned} Y_{n,1}(\xi) &= P_{n0}(\cos\theta) \\ Y_{n,2}(\xi) &= P_{n1}(\cos\theta) \cos\lambda \\ Y_{n,3}(\xi) &= P_{n1}(\cos\theta) \sin\lambda \\ &\vdots \\ Y_{n,2n}(\xi) &= P_{nn}(\cos\theta) \cos(n\lambda) \\ Y_{n,2n+1}(\xi) &= P_{nn}(\cos\theta) \sin(n\lambda) \end{aligned} \quad (8.7)$$

$$(\xi=(\xi_1,\xi_2,\xi_3)^T, \xi_1=\sin\theta\cos\lambda, \xi_2=\sin\theta\sin\lambda, \xi_3=\cos\theta) .$$

is a maximal linearly independent system of (surface) spherical harmonics of order  $n$ .

Our aim now is to point out some relations between

these two systems of linearly independent spherical harmonics.

As maximal linearly independent systems, both are bases in the space  $S_n$  of (surface) spherical harmonics of order  $n$ . Thus each element of the first basis may be represented by a linear combination of the second basis, and vice versa.

In particular, we have

$$Y_{n,j}(\xi) = \sum_{\mu=0}^{2n} g_{\mu}^j H_{n,2n+1-\mu}(\xi), \quad \xi \in \Omega \quad (8.8)$$

with suitable coefficients

$$g_0^j, \dots, g_{2n}^j.$$

This means that the system  $\{Y_{n,j}\}_{j=1,\dots,2n+1}$  is representable as sum of monomials  $\xi^\alpha$ :

$$Y_{n,j}(\xi) = \sum_{[\alpha]=n} E_{\alpha}^j \xi^{\alpha}, \quad \xi \in \Omega \quad (8.9)$$

Therefore, by comparison of (8.6), (8.8) and (8.9), we obtain

$$\begin{aligned} Y_{n,j}(\xi) &= \sum_{[\alpha]=n} E_{\alpha}^j \xi^{\alpha} \\ &= \sum_{\mu=0}^{2n} g_{\mu}^j \sum_{[\alpha]=n} C_{\alpha}^{2n+1-\mu} \xi^{\alpha} \\ &= \sum_{[\alpha]=n} \sum_{\mu=0}^{2n} g_{\mu}^j C_{\alpha}^{2n+1-\mu} \xi^{\alpha}, \end{aligned} \quad (8.10)$$

i. e.:

$$\sum_{\mu=0}^{2n} g_{\mu}^j c_{\alpha}^{2n+1-\mu} = E_{\alpha}^j, \quad (8.11)$$

$$([\alpha]=n, j=1, \dots, 2n+1).$$

In vectorial form this yields

$$c g = e, \quad (8.12)$$

where c, e, g are given as follows

$$c = \left[ \begin{array}{cccc} c_{n,0,0}^1 & c_{n,0,0}^2 & \dots & c_{n,0,0}^{2n+1} \\ c_{n-1,1,0}^1 & c_{n-1,1,0}^2 & & c_{n-1,1,0}^{2n+1} \\ c_{n-1,0,1}^1 & c_{n-1,0,1}^2 & \dots & c_{n-1,0,1}^{2n+1} \\ \vdots & \vdots & & \vdots \\ c_{2,n-2,0}^1 & c_{2,n-2,0}^2 & \dots & c_{2,n-2,0}^{2n+1} \\ \vdots & \vdots & & \vdots \\ c_{2,0,n-2}^1 & c_{2,0,n-2}^2 & \dots & c_{2,0,n-2}^{2n+1} \\ \hline c_{1,n-1,0}^1 & 0 & \dots & 0 \\ 0 & c_{1,n-2,1}^2 & \dots & \vdots \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \\ & & & 0 \end{array} \right] \begin{array}{l} \left. \begin{array}{l} (n+1) \\ 2 \end{array} \right\} \\ (2n+1) \\ \left. \begin{array}{l} 2n+1 \end{array} \right\} \end{array}$$

2n+1



According to the ordering of the multiindices introduced in chapter 6 we are able to rewrite the matrices  $c$  and  $e$  as follows:

$$c = \begin{pmatrix} c_1^1 & c_1^2 & \dots & c_1^{2n+1} \\ c_2^1 & c_2^2 & \dots & c_2^{2n+1} \\ c_3^1 & c_3^2 & \dots & c_3^{2n+1} \\ \vdots & \vdots & & \vdots \\ c_{M-(3n-1)}^1 & c_{M-(3n-1)}^2 & \dots & c_{M-(3n-1)}^{2n+1} \\ \vdots & \vdots & & \vdots \\ c_{M-(2n-1)}^1 & c_{m-(2n+1)}^2 & \dots & c_{M-(2n+1)}^{2n+1} \\ \hdashline c_{M-(2n)}^1 & 0 & \dots & 0 \\ & \ddots & & \vdots \\ 0 & c_{M-(2n-1)}^2 & \dots & 0 \\ & \ddots & & \vdots \\ 0 & & \ddots & 0 \\ 0 & & & c_M^{2n+1} \end{pmatrix}$$

and

$$e = \begin{pmatrix} E_1^1 & E_1^2 & \dots & E_1^{2n+1} \\ E_2^1 & E_2^2 & \dots & E_2^{2n+1} \\ E_3^1 & E_3^2 & \dots & E_3^{2n+1} \\ \vdots & \vdots & & \vdots \\ E_{M-(3n-1)}^1 & E_{M-(3n-1)}^2 & \dots & E_{M-(3n-1)}^{2n+1} \\ \vdots & \vdots & & \vdots \\ E_{M-(2n+1)}^1 & E_{M-(2n+1)}^2 & \dots & E_{M-(2n+1)}^{2n+1} \\ \hline E_{M-(2n)}^1 & E_{M-(2n)}^2 & \dots & E_{M-(2n)}^{2n+1} \\ E_{M-(2n-1)}^1 & E_{M-(2n-1)}^2 & \dots & E_{M-(2n-1)}^{2n+1} \\ \vdots & \vdots & & \vdots \\ E_{M-(n+1)}^1 & E_{M-(n+1)}^2 & \dots & E_{M-(n+1)}^{2n+1} \\ E_{M-n}^1 & E_{M-n}^2 & \dots & E_{M-n}^{2n+1} \\ \vdots & \vdots & & \vdots \\ E_M^1 & E_M^2 & \dots & E_M^{2n+1} \end{pmatrix}.$$

This leads to the linear equations

$$\sum_{\mu=0}^{2n} g_{\mu}^j c_v^{2n+1-\mu} = E_v^j. \quad (8.13)$$

( $v=1, \dots, M=\binom{n+2}{2}$ )



From the special structure of the matrix  $c$  we deduce that

$$\begin{aligned}
 g_{2n}^j &= \frac{E_{M-(2n)}^j}{C_{M-(2n)}^1}, & j &= 1, \dots, 2n+1, \\
 &\vdots & & \\
 g_{n+1}^j &= \frac{E_{M-(n+1)}^j}{C_{M-(n+1)}^n}, & j &= 1, \dots, 2n+1, \\
 &\vdots & & \\
 g_n^j &= \frac{E_{M-n}^j}{C_{M-n}^{n+1}}, & j &= 1, \dots, 2n+1, \\
 &\vdots & & \\
 g_0^j &= \frac{E_M^j}{C_M^{2n+1}}, & j &= 1, \dots, 2n+1,
 \end{aligned}$$

i. e.:

$$g_\mu^j = E_{M-\mu}^j \cdot \{C_{M-\mu}^{(2n+1)-\mu}\}^{-1} \quad (8.14)$$

( $\mu = 0, \dots, 2n$ )

In order to calculate these values we need the coefficients  $E_{M-(2n)}^j, \dots, E_M^j, j = 1, \dots, 2n+1$ .

To this end we recall to our mind that

$$P_{nm}(\cos\theta) = (\sin\theta)^{m/2} \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-m-2k)!} (\cos\theta)^{n-m-2k}.$$

Furthermore, it is well-known that

$$\begin{aligned} \cos(m\lambda) &= \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k m!}{(2k)!(m-2k)!} (\cos\lambda)^{m-2k} (\sin\lambda)^{2k} \\ \sin(m\lambda) &= \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^k m!}{(2k+1)!(m-2k-1)!} (\cos\lambda)^{m-2k-1} (\sin\lambda)^{2k+1} \end{aligned} \quad (8.15)$$

Thus, an elementary computation gives

$$\begin{aligned} P_{nm}(\cos\theta) \cos(m\lambda) &= \frac{1}{|x|^n} \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{k+l} \frac{m! (2n-2k)!}{2^n k! (n-k)! (n-m-2k)! (2l)!(m-2l)!} |x|^{2k} x_1^{m-2l} x_2^{2l} x_3^{n-m-2k} \end{aligned} \quad (8.16)$$

and

$$\begin{aligned} P_{nm}(\cos\theta) \sin(m\lambda) &= \frac{1}{|x|^n} \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{k+l} \frac{m! (2n-2k)!}{2^n k! (n-k)! (n-m-2k)! (2l+1)!(m-2l-1)!} |x|^{2k} x_1^{m-2l-1} x_2^{2l+1} x_3^{n-m-2k} \end{aligned}$$

According to the polynomial theorem we obtain

$$|x|^{2k} = (x_1^2 + x_2^2 + x_3^2)^k = \sum_{[\alpha]=k} \frac{k!}{\alpha!} x^{2\alpha} \quad (8.17)$$

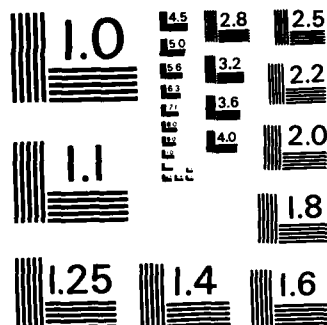
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Thus we find after an easy calculation

$$P_{nm}(\cos\theta) \cos(m\lambda)$$

$$= \sum_{k=0}^{\left[\frac{n-m}{2}\right]} \sum_{l=0}^{\left[\frac{m}{2}\right]} \beta_k^{n,m} \gamma_l^m \xi_1^{m-2l} \xi_2^{2l} \xi_3^{n-m} x_k$$

and

(8.18)

$$P_{nm}(\cos\theta) \sin(m\lambda)$$

$$= \sum_{k=0}^{\left[\frac{n-m}{2}\right]} \sum_{l=0}^{\left[\frac{m-1}{2}\right]} \beta_k^{n,m} \sigma_l^m \xi_1^{m-1-2l} \xi_2^{2l+1} \xi_3^{n-m} x_k,$$

where we have used the abbreviations

$$\beta_k^{n,m} = (-1)^k \frac{m!}{2^n} \frac{(2n-2k)!}{(n-k)!(n-m-2k)!}$$

$$\gamma_l^m = (-1)^l \frac{1}{(2l)!(m-2l)!}$$

$$\sigma_l^m = (-1)^l \frac{1}{(2l+1)!(m-2l-1)!}$$

and

$$x_k = \sum_{\tau=0}^k \sum_{v=0}^{k-\tau} \frac{1}{\tau!v!(k-\tau-v)!} \xi_1^{2\tau} \xi_2^{2v} \xi_3^{-2\tau-2v}. \quad (8.19)$$

We write out only those terms to be needed for the determination of the values (8.14):

$$P_{nm}(\cos\theta) \cos(m\lambda)$$

$$= \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} \beta_k^{n,m} \gamma_{\lfloor \frac{m}{2} \rfloor}^m \sum_{\mu=0}^k \frac{1}{\mu!(k-\mu)!} \xi_1^{m-2\lfloor \frac{m}{2} \rfloor} \xi_2^{2\mu+2\lfloor \frac{m}{2} \rfloor} \xi_3^{2\mu+2\lfloor \frac{m}{2} \rfloor} \xi_3^{n-m-2\mu} + \dots$$

$$(m = 0, 1, \dots, n)$$

resp.

(8.19)

$$P_{nm}(\cos\theta) \sin(m\lambda)$$

$$= \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} \beta_k^{n,m} \sigma_{\lfloor \frac{m-1}{2} \rfloor}^m \sum_{\mu=0}^k \frac{1}{\mu!(k-\mu)!} \xi_1^{m-1-2\lfloor \frac{m-1}{2} \rfloor} \xi_2^{2\lfloor \frac{m-1}{2} \rfloor+2\mu} \xi_3^{n-m-2\mu} + \dots$$

$$(m = 1, \dots, 2)$$

It is easy to see that

$$\begin{aligned} & \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} \beta_k^{n,m} \gamma_{\lfloor \frac{m}{2} \rfloor}^m \sum_{\mu=0}^k \frac{1}{\mu!(k-\mu)!} \xi_1^{m-2\lfloor \frac{m}{2} \rfloor} \xi_2^{2\mu+2\lfloor \frac{m}{2} \rfloor} \xi_3^{n-m-2\mu} \\ &= \sum_{v=0}^{\lfloor \frac{n-m}{2} \rfloor} \left[ \sum_{k=v}^{\lfloor \frac{n-m}{2} \rfloor} \beta_k^{n,m} \frac{1}{v!(k-v)!} \right] \gamma_{\lfloor \frac{m}{2} \rfloor}^m \xi_1^{m-2\lfloor \frac{m}{2} \rfloor} \xi_2^{2v+2\lfloor \frac{m}{2} \rfloor} \xi_3^{n-m-2v} \end{aligned}$$

$$(m = 0, (1), n)$$

and

(8.20)

$$\begin{aligned}
 & \sum_{k=0}^{\left[\frac{n-m}{2}\right]} \beta_k^{n,m} \sigma_{\left[\frac{m-1}{2}\right]}^m \sum_{\mu=0}^k \frac{1}{\mu!(k-\mu)!} \xi_1^{m-1-2\left[\frac{m-1}{2}\right]} \xi_2^{2\left[\frac{m-1}{2}\right]+2\mu} \xi_3^{n-m-2\mu} \\
 = & \sum_{v=0}^{\left[\frac{n-m}{2}\right]} \left[ \sum_{k=v}^{\left[\frac{n-m}{2}\right]} \beta_k^{n,m} \frac{1}{v!(k-v)!} \right] \sigma_{\left[\frac{m-1}{2}\right]}^m \xi_1^{m-1-2\left[\frac{m-1}{2}\right]} \xi_2^{2v-2\left[\frac{m-1}{2}\right]} \xi_3^{n-m-2v} \\
 & (m = 1, (1), n)
 \end{aligned}$$

Consequently we find

$m = 0$ :

$$Y_{n,1}(\xi) = P_{n0}(\cos \theta) = \sum_{\mu=0}^{2n} g_{\mu}^1 H_{n,2n+1-\mu}(\xi)$$

with

$$g_{\mu}^1 = \begin{cases} \gamma_0^0 \sum_{k=\frac{\mu}{2}}^{\left[\frac{n}{2}\right]} \beta_k^{n,0} \frac{1}{\left(\frac{\mu}{2}\right)!(k-\frac{\mu}{2})!} \{C_{M-\mu}^{(2n+1)-\mu}\}^{-1} \\ \text{for } \mu = 0, (2), n - \left[1 + \frac{(-1)^{n+1}}{2}\right] \\ 0 \quad \text{otherwise} \end{cases}$$

$1 \leq m \leq n$ ;  $m$  even:

$$Y_{n,2m}(\xi) = P_{nm}(\cos \xi) \cos(m\lambda) = \sum_{\mu=0}^{2n} g_{\mu}^{2m} H_{n,2n+1-\mu}(\xi)$$

with

$$g_{\mu}^{2m} = \begin{cases} \gamma_{\frac{m}{2}}^m \sum_{k=\frac{\mu-m}{2}}^{\lfloor \frac{n-m}{2} \rfloor} \beta_k^{n,m} \frac{1}{(\frac{\mu-m}{2})! (k - \frac{\mu-m}{2})!} \{C_{M-\mu}^{(2n+1)-\mu}\}^{-1} \\ \text{for } \mu = m, (2), n - [1 + \frac{(-1)^{n+1}}{2}] \\ 0 \quad \text{otherwise} \end{cases}$$

$1 \leq m \leq n; m \text{ odd:}$

$$Y_{n,2m}(\xi) = P_{nm}(\cos\theta) \cos(m\lambda) = \sum_{\mu=0}^{2n} g_{\mu}^{2m} H_{n,2n+1-\mu}(\xi)$$

with

$$g_{\mu}^{2m} = \begin{cases} \gamma_{\frac{m-1}{2}}^m \sum_{k=\frac{\mu-m-n}{2}}^{\lfloor \frac{n-m}{2} \rfloor} \beta_k^{n,m} \frac{1}{(\frac{\mu-m-n}{2})! (k - \frac{\mu-m-n}{2})!} \{C_{M-\mu}^{(2n+1)-\mu}\}^{-1} \\ \text{for } \mu = n+m, (2), n - [1 + \frac{(-1)^n}{2}] \\ 0 \quad \text{otherwise} \end{cases}$$

$1 \leq m \leq n; m \text{ even:}$

$$Y_{n,2m+1}(\xi) = P_{nm}(\cos\theta) \sin(m\lambda) = \sum_{\mu=0}^{2n} g_{\mu}^{2m+1} H_{n,2n+1-\mu}(\xi)$$

with



$$g_{\mu}^{2m+1} = \begin{cases} \sigma_{\frac{m-2}{2}}^m \sum_{k=\frac{\mu-m-n}{2}}^{\lfloor \frac{n-m}{2} \rfloor} \beta_k^{n,m} \frac{1}{(\frac{\mu-m-n}{2})! (k - \frac{\mu-m-n}{2})!} \{C_{M-\mu}^{(2n+1)-\mu}\}^{-1} & \text{for } \mu = n+m, (2), 2n - [1 + \frac{(-1)^{n+1}}{2}] \\ 0 & \text{otherwise} \end{cases}$$

$1 \leq m \leq n; m \text{ odd:}$

$$Y_{n,2m+1}(\xi) = P_{nm}(\cos \theta) \sin(m\lambda) = \sum_{\mu=0}^{2n} g_{\mu}^{2m+1} H_{n,2n+1-\mu}(\xi)$$

with

$$g_{\mu}^{2m+1} = \begin{cases} \sigma_{\frac{m-1}{2}}^m \sum_{k=\frac{\mu-m}{2}}^{\lfloor \frac{n-m}{2} \rfloor} \beta_k^{n,m} \frac{1}{(\frac{\mu-m}{2})! (k - \frac{\mu-m}{2})!} \{C_{M-\mu}^{2n+1-\mu}\}^{-1} & \text{for } \mu = m, (2), n - [1 + \frac{(-1)^n}{2}] \\ 0 & \text{otherwise} \end{cases}$$

Furthermore we get (cf. (0.5))

$$Y_{n,1}^*(\xi) = \sqrt{2n+1} \sum_{\mu=0}^{2n} g_{\mu}^1 H_{n,2n+1-\mu}(\xi)$$

$$Y_{n,j}^*(\xi) = \sqrt{2(2n+1) \frac{(n-\lfloor \frac{j}{2} \rfloor)!}{(n+\lfloor \frac{j}{2} \rfloor)!}} \sum_{\mu=0}^{2n} g_{\mu}^j H_{n,2n+1-\mu}(\xi) .$$

(j = 2, (1), 2n+1)

Therefore the addition theorem may be rewritten as follows:

Let  $\{H_{n,j}\}_{j=1,\dots,2n+1}$  be the sequence of linearly independent homogeneous harmonic polynomials of degree  $n$  generated by exact computation (cf. chapter 6). Then, for any two  $\xi, \eta \in \Omega$ ,

$$\begin{aligned} P_n(\xi\eta) &= \sum_{\mu=0}^{2n} \sum_{v=0}^{2n} g_{\mu}^1 g_v^1 H_{n,2n+1-\mu}(\xi) H_{n,2n+1-v}(\eta) \\ &+ 2 \sum_{j=2}^{2n+1} \frac{(n-\lfloor \frac{j}{2} \rfloor)!}{(n+\lfloor \frac{j}{2} \rfloor)!} \sum_{\mu=0}^{2n} \sum_{v=0}^{2n} g_{\mu}^j g_v^j H_{n,2n+1-\mu}(\xi) H_{n,2n+1-v}(\eta). \end{aligned}$$

#### Examples:

We demonstrate some relations between the system (8.6) and (8.7) for the degrees  $n = 3, 5$ .

**Example 1:**  $n = 3$

$$P_{30}(\cos\theta) = \xi_3^3 - \frac{3}{2} \xi_1^2 \xi_3 - \frac{3}{2} \xi_2^2 \xi_3 = Y_{3,1}(\xi)$$

$$P_{31}(\cos\theta) \cos\lambda = 6 \xi_1 \xi_3^2 - \frac{3}{2} \xi_1^3 - \frac{3}{2} \xi_1 \xi_2^2 = Y_{3,2}(\xi)$$

$$P_{31}(\cos\theta) \sin\lambda = 3 \xi_2 \xi_3^2 - \frac{3}{4} \xi_1^2 \xi_2 - \frac{3}{4} \xi_2^3 = Y_{3,3}(\xi)$$

$$P_{32}(\cos\theta) \cos(2\lambda) = 15 \xi_1^2 \xi_3 - 15 \xi_2^2 \xi_3 = Y_{3,4}(\xi)$$

$$P_{32}(\cos\theta) \sin(2\lambda) = 30 \xi_1 \xi_2 \xi_3 = Y_{3,5}(\xi)$$

$$P_{33}(\cos\theta) \cos(3\lambda) = 15 \xi_1^3 - 45 \xi_1 \xi_2^2 = Y_{3,6}(\xi)$$

$$P_{33}(\cos\theta) \sin(3\lambda) = 45 \xi_1^2 \xi_2 - 15 \xi_2^3 = Y_{3,7}(\xi)$$

$$(\xi = (\xi_1, \xi_2, \xi_3)^T, \xi_1 = \sin\theta \cos\lambda, \xi_2 = \sin\theta \sin\lambda, \xi_3 = \cos\theta)$$

Thus, we have

$$c = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ \hline -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$e = \begin{bmatrix} 0 & -\frac{3}{2} & 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & 45 \\ -\frac{3}{2} & 0 & 0 & 15 & 0 & 0 & 0 \\ \hline 0 & -\frac{3}{2} & 0 & 0 & 0 & -45 & 0 \\ 0 & 0 & 0 & 0 & 30 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 & 0 & 0 & -15 \\ -\frac{3}{2} & 0 & 0 & -15 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This yields

$$g = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 15 \\ -\frac{3}{2} & 0 & 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consequently, we find

$$\begin{aligned}
 Y_{3,1}(\xi) &= -\frac{3}{2} H_{3,5}(\xi) - 1 \cdot H_{3,7}(\xi) \\
 Y_{3,2}(\xi) &= \frac{1}{2} H_{3,1}(\xi) - 2 \cdot H_{3,3}(\xi) \\
 Y_{3,3}(\xi) &= \frac{3}{4} H_{3,4}(\xi) - 3 \cdot H_{3,6}(\xi) \\
 Y_{3,4}(\xi) &= 15 H_{3,5}(\xi) \\
 Y_{3,5}(\xi) &= -30 H_{3,2}(\xi) \\
 Y_{3,6}(\xi) &= 15 H_{3,1}(\xi) \\
 Y_{3,7}(\xi) &= 15 H_{3,4}(\xi)
 \end{aligned}$$

For example, we have

$$\begin{aligned}
 Y_{3,7}(\xi) &= g_3^7 H_{3,4}(\xi) \\
 &= \sigma_1^3 \beta_0^{3,3} \{C_{M-3}^4\}^{-1} H_{3,4}(\xi)
 \end{aligned}$$

(Observe that  $C_{M-3}^4 = -1$ )

$$= 15 H_{3,4}(\xi) .$$

Moreover, we get

$$\begin{aligned}
 Y_{3,1}^*(\xi) &= \sqrt{7} \left( -\frac{3}{2} H_{3,5}(\xi) - H_{3,7}(\xi) \right) \\
 Y_{3,2}^*(\xi) &= \sqrt{\frac{7}{6}} \left( \frac{1}{2} H_{3,1}(\xi) - 2H_{3,3}(\xi) \right) \\
 Y_{3,3}^*(\xi) &= \sqrt{\frac{7}{6}} \left( \frac{3}{4} H_{3,4}(\xi) - 3H_{3,6}(\xi) \right) \\
 Y_{3,4}^*(\xi) &= \sqrt{\frac{7}{60}} \left( 15 H_{3,5}(\xi) \right) \\
 Y_{3,5}^*(\xi) &= \sqrt{\frac{7}{60}} \left( -30 H_{3,2}(\xi) \right) \\
 Y_{3,6}^*(\xi) &= \sqrt{\frac{7}{360}} \left( 15 H_{3,1}(\xi) \right) \\
 Y_{3,7}^*(\xi) &= \sqrt{\frac{7}{360}} \left( 15 H_{3,4}(\xi) \right) .
 \end{aligned}$$

**Example 2:**  $n = 5$

$$\begin{aligned}
 Y_{5,1}(\xi) &= g_0^1 H_{5,11}(\xi) + g_2^1 H_{5,9}(\xi) + g_4^1 H_{5,7}(\xi) \\
 Y_{5,2}(\xi) &= g_6^2 H_{5,5}(\xi) + g_8^2 H_{5,3}(\xi) + g_{10}^2 H_{5,1}(\xi) \\
 Y_{5,3}(\xi) &= g_1^3 H_{5,10}(\xi) + g_3^3 H_{5,8}(\xi) + g_5^3 H_{5,6}(\xi) \\
 Y_{5,4}(\xi) &= g_2^4 H_{5,9}(\xi) + g_4^4 H_{5,7}(\xi) \\
 Y_{5,5}(\xi) &= g_7^5 H_{5,4}(\xi) + g_9^5 H_{5,2}(\xi) \\
 Y_{5,6}(\xi) &= g_8^6 H_{5,3}(\xi) + g_{10}^6 H_{5,1}(\xi) \\
 Y_{5,7}(\xi) &= g_3^7 H_{5,8}(\xi) + g_5^7 H_{5,6}(\xi) \\
 Y_{5,8}(\xi) &= g_4^8 H_{5,7}(\xi)
 \end{aligned}$$

$$Y_{5,9}(\xi) = g_9^9 H_{5,2}(\xi)$$

$$Y_{5,10}(\xi) = g_{10}^{10} H_{5,1}(\xi)$$

$$Y_{5,11}(\xi) = g_5^{11} H_{5,6}(\xi) \quad .$$

We omit the explicit calculation of the coefficients.

9. Representation of External Gravitational Potential by Means of Spherical Harmonics (Exterior Dirichlet's Problem)

A surface  $S \subset R^3$  will be called regular, if it satisfies the following properties:

- (i)  $S$  divides three-dimensional Euclidean space  $R^3$  uniquely into the bounded region  $E = E_i$  (inner space) and the unbounded region  $E_e$  (outer space) defined by  $E_e = R^3 - \bar{E}_e$ ,  $\bar{E}_e = E_e \cup S$ . The origin belongs to  $E_i$ .
- (ii)  $S$  is a closed and compact surface with no double points.
- (iii)  $S$  is a  $C^{(2)}$  - surface.

From the definition it is clear that all (geometrically relevant) earth models are included. Regular surfaces are for example sphere, ellipsoid, spheroid, telluroid or real (regular) earth's surface).

Let  $V$  be a function satisfying the following properties:

- (i)  $V$  is continuous in  $\bar{E}_e = E_e \cup S$  and twice continuously differentiable in  $E_e$



(ii)  $V$  is harmonic in  $E_e$ :

$$\Delta V(x) = 0 \quad \text{for } x \in E_e$$

(iii)  $V$  is regular at infinity:

$$\begin{aligned} |V(x)| &= O\left(\frac{1}{|x|}\right) \\ |\nabla V(x)| &= O\left(\frac{1}{|x|^2}\right) \end{aligned} \quad |x| \rightarrow \infty$$

We consider the countably infinite system of outer harmonics

$$\left\{ \frac{1}{|x|^{2n+1}} H_{n,j}(x) \right\}_{\substack{n=0,1,\dots \\ j=1,\dots,2n+1}}, \quad (9.1)$$

where, for each  $n$ , the sequence

$$\{H_{n,j}\}_{j=1,\dots,2n+1} \quad (9.2)$$

forms a linearly independent system of homogeneous harmonic polynomials of degree  $n$  (cf. chapter 6).

Then there exists a system of outer harmonics

$$\{H_{n,j}^*\}_{\substack{n=0,1,\dots \\ j=1,\dots,2n+1}} \quad (9.3)$$

orthonormal with respect to the  $L^2$ -inner product:

$$\begin{aligned} (H_{n,j}^*, H_{n,1}^*)_{L^2(S)} &= \frac{1}{\|S\|} \int_S H_{n,j}^*(y) H_{n,1}^*(y) d\omega \\ &= \delta_{j1} . \end{aligned} \quad (9.4)$$

( $\|S\|$ : total surface of  $S$ ). According to the expansion theorem developed by the author (1983) the potential  $V$  can be represented by the series

$$V(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left\{ \frac{1}{\|S\|} \int_S V(y) H_{n,j}^*(y) d\omega \right\} H_{n,j}^*(x) , \quad (9.5)$$

where the numbers

$$(V, H_{n,j}^*)_{L^2(S)} = \frac{1}{\|S\|} \int_S V(y) H_{n,j}^*(y) d\omega$$

are the Fourier (or orthogonal) coefficients of  $V$  on  $S$  with respect to the system  $\{H_{n,j}^*\}_{n=0,1,\dots, j=1,\dots,2n+1}$ .

More explicitly this reads: given an error bound  $\varepsilon > 0$ , then there exists an integer  $N = N(\varepsilon)$  such that

$$|V(x) - V_N(x)| \leq \varepsilon \quad (9.6)$$

with

$$V_N(x) = \sum_{n=0}^N \sum_{j=1}^{2n+1} \left\{ \frac{1}{\|S\|} \int_S V(y) H_{n,j}^*(y) d\omega \right\} H_{n,j}^*(x) \quad (9.7)$$

holds for all  $x \in G \subset E_e$  and  $\text{dist}(G, S) \geq \rho > 0$ . In each compact subset  $K \subset E_e$  with  $\text{dist}(K, S) \geq \rho > 0$  the convergence is uniform

$$\sup_{x \in K} |V(x) - V_N(x)| \leq \varepsilon .$$

The series guarantees ordinary pointwise convergence, in fact, in the (whole) outer space  $E_e$ .

In the context of boundary-value problems of potential theory

$$V_N(x) = \sum_{n=0}^N \sum_{j=1}^{2n+1} \left\{ \frac{1}{\|S\|} \int_S f(y) H_{n,j}^*(y) d\omega \right\} H_{n,j}^*(x)$$

may be interpreted as approximation to the exterior Dirichlet's problem

$$\Delta V(x) = 0 \quad \text{for } x \in E_e$$

$$|V(x)| = O\left(\frac{1}{|x|}\right), \quad |\nabla V(x)| = O\left(\frac{1}{|x|^2}\right) \\ (|x| \rightarrow \infty)$$

$$V(x) = f(x) \quad \text{for } x \in S \quad (f \in C^{(0)}(S))$$

by the orthogonal expansion in terms of outer harmonics  $H_{n,j}^*$  of order  $\leq N$ .

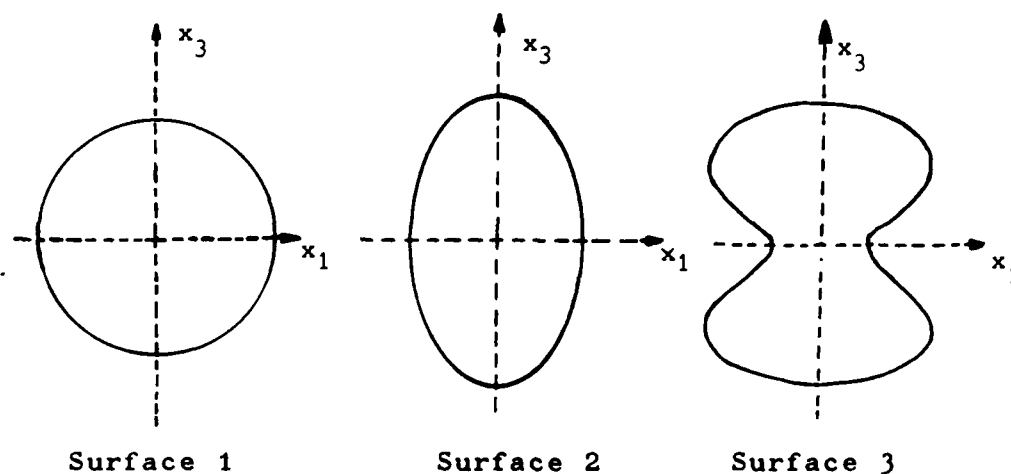
When our expansion theorem is formulated especially for a sphere  $S$  about the origin with radius  $R$ , we are led to the classical results (cf. Introduction).

Approximation by outer harmonics (Dirichlet's problem)	S regular surface	S Sphere about the origin with radius R
linearly independent system	$H_{n,j}$	$H_{n,j}$
approximation	$V_N(x) = \sum_{n=0}^N \sum_{j=1}^{2n+1} C_{n,j} H_{n,j}^*(x)$	$V_N(x) = \sum_{n=0}^N \sum_{j=1}^{2n+1} C_{n,j} \left(\frac{R}{ x }\right)^{n+1} S_{n,j}^*(\xi)$ $(x =  x  \xi, \xi \in \Omega)$
orthogonal coefficients	$C_{n,j} = \frac{1}{\ S\ } \int_S V(y) H_{n,j}^*(y) d\omega(y)$	$C_{n,j} = \frac{1}{4\pi R^n} \int_{\Omega} V(R\eta) S_{n,j}^*(\eta) d\omega(\eta)$
Region of (pointwise) convergence ( $N \rightarrow \infty$ )	$\{x \mid x \in E_e\}$	$\{x \mid  x  > R\}$

**Examples:** In order to get an impression of the method of computing a potential  $V$  by the the (generalized) Fourier procedure proposed here we discuss some sample examples.

(i) **Boundaries**

Three boundaries will be used in the examples, their cross-sections in the  $(x_1, x_3)$ -plane are shown in the figures



The first surface (Surface 1) is a sphere given by  $x \in \mathbb{R}^3$ ,  $x = (x_1, x_2, x_3)^T$  with

$$\begin{aligned} x_1 &= \frac{3}{2} \sin\theta \cos\lambda \\ x_2 &= \frac{3}{2} \sin\theta \sin\lambda \\ x_3 &= \frac{3}{2} \cos\theta \end{aligned} \quad \left[ \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \lambda < 2\pi \end{array} \right]$$

The second surface (Surface 2) is an ellipsoid given by

$$\begin{aligned} x_1 &= 1 \sin \theta \cos \lambda \\ x_2 &= \frac{3}{2} \sin \theta \sin \lambda \\ x_3 &= 2 \cos \theta \end{aligned} \quad \left[ \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \lambda < 2\pi \end{array} \right]$$

The third surface (Surface 3) is given by

$$\begin{aligned} x_1 &= 2 r(\theta) \sin \theta \cos \lambda \\ x_2 &= 3 r(\theta) \sin \theta \sin \lambda \\ x_3 &= r(\theta) \cos \theta \end{aligned} \quad \left[ \begin{array}{l} 0 \leq \theta \leq \pi \\ 0 \leq \lambda < 2\pi \end{array} \right]$$

with

$$r(\theta) = \{ \cos(2\theta) + (\frac{11}{10} - \sin^2(2\theta))^{1/2} \}^{1/2}.$$

## (ii) Potentials

We want to give expansions for the following two potentials:

$$v^{(1)}(x) = \frac{1}{|x|} \left( e^{\frac{x_1}{|x|^2}} \cos \left( \frac{x_2}{|x|^2} \right) + e^{\frac{x_3}{|x|^2}} \sin \left( \frac{x_1}{|x|^2} \right) \right)$$

$$v^{(2)}(x) = \frac{1}{|x|} \frac{1}{\left[ \left( \frac{x_1}{|x|^2} - 5 \right)^2 + \left( \frac{x_2}{|x|^2} - 4 \right)^2 + \left( \frac{x_3}{|x|^2} - 3 \right)^2 \right]^{1/2}}$$

$$(x^T = (x_1, x_2, x_3))$$

The potentials  $v^{(1)}$  ,  $v^{(2)}$  are the Kelvin transforms of the functions given by

$$e^{x_1} \cos(x_2) + e^{x_3} \sin x_1 ,$$

$$[(x_1-5)^2 + (x_2-4)^2 + (x_3-3)^2]^{-1/2}$$

respectively. The same functions have been studied by K. E. Atkinson (1980/1982) using integral equation methods.

### (iii) Numerical Method

The numerical computation was done via the so-called normal equations (using Cholesky's method) as described by the author (cf. W. Freeden (1983), chapt. 11).

The inner products (integrals) occurring in the normal equations were computed by the approximate sums

$$\int_S f(x) d\omega \approx \frac{1}{A} \sum_{m_1, m_2} f(x_{m_1, m_2})$$

where  $\{x_{m_1, m_2}\}$  represents a (nearly) uniform distribution of nodes on  $S$  with total number  $A$ .

In our examples the subdivision  $\{x_{m_1, m_2}\}$  on  $S$  is generated by polar coordinates

$$x_{m_1, m_2} = |x_{m_1, m_2}| \xi_{m_1, m_2},$$

$$\xi_{m_1, m_2} = (\sin \theta_{m_1} \cos \lambda_{m_1, m_2}, \sin \theta_{m_1} \sin \lambda_{m_1, m_2}, \cos \theta_{m_1})^T$$

with

$$\theta_{m_1} = m_1 \frac{\pi}{n_1} - \frac{\pi}{2n_1}$$

$$\lambda_{m_1, m_2} = m_2 \frac{2\pi}{n_2 \sin(\theta_{m_1})} - \frac{\pi}{n_2 \sin(\theta_{m_1})}$$

$$\left[ \begin{array}{l} 0 < m_1 \leq n_1, \quad 0 < m_2 \leq n_2 \sin \theta_{m_1} \\ n_1, n_2: \text{ (fixed) positive integers} \end{array} \right].$$

We choose especially  $n_1 = 21$ ,  $n_2 = 42$ . Then the total sum of nodes on  $S$  is  $A = 562$ .

#### (iv) Error estimates

We set

$$E_N^{(1)} = v^{(1)} - v_N^{(1)}$$

$$E_N^{(2)} = v^{(2)} - v_N^{(2)}$$

We give an impression of the error for a set of selected points.



**Table 1: Expansion of  $V^{(1)}$  (Surface 1)**

$x_1$	$x_2$	$x_3$	$V^{(1)}$	$E_5^{(1)}$	$E_6^{(1)}$
1.0	1.5	2.0	0.4845 E+0	-0.31 E-6	-0.34 E- 8
1.5	0.0	0.0	0.1711 E+1	-0.82 E-4	-0.79 E- 6
3.0	0.0	0.0	0.5743 E+0	-0.64 E-6	-0.34 E- 8
0.0	2.0	0.0	0.4388 E+0	0.11 E-4	-0.48 E- 7
0.0	4.0	0.0	0.2422 E+0	0.84 E-7	0.12 E-11
0.0	0.0	2.5	0.4000 E+0	0.53 E-8	0.92 E- 9
0.0	0.0	5.0	0.2000 E+0	0.76 E-9	0.15 E- 9

**Table 2: Expansion of  $V^{(1)}$  (Surface 2)**

$x_1$	$x_2$	$x_3$	$V^{(1)}$	$E_5^{(1)}$	$E_6^{(1)}$
1.0	1.5	2.0	0.4845 E+0	-0.91 E-5	-0.76 E- 6
1.5	0.0	0.0	0.1711 E+1	-0.19 E-4	-0.72 E- 6
3.0	0.0	0.0	0.5743 E+0	-0.13 E-5	-0.99 E- 7
0.0	2.0	0.0	0.4388 E+0	0.54 E-4	-0.54 E- 6
0.0	4.0	0.0	0.2422 E+0	0.35 E-5	-0.24 E- 7
0.0	0.0	2.5	0.4000 E+0	-0.74 E-6	0.36 E- 8
0.0	0.0	5.0	0.2000 E+0	0.19 E-6	-0.95 E- 9

Table 3: Expansion of  $v^{(1)}$  (Surface 3)

$x_1$	$x_2$	$x_3$	$v^{(1)}$	$E_5^{(1)}$	$E_6^{(1)}$
0.0	0.0	2.0	0.5000 E+0	-0.18 E-3	-0.66 E-6
0.0	0.0	4.0	0.2500 E+0	-0.24 E-3	0.29 E-6
0.7	0.0	0.0	0.7375 E+0	0.82 E-3	-0.98 E-3
1.4	0.0	0.0	0.1927 E+1	-0.35 E-3	-0.58 E-3
0.0	1.0	0.0	0.5403 E+0	0.35 E-2	-0.52 E-4
0.0	2.0	0.0	0.4388 E+0	-0.12 E-2	0.17 E-4
1.0	1.5	0.75	0.7765 E+0	-0.23 E-2	-0.39 E-3
2.0	3.0	1.5	0.3233 E+0	-0.57 E-3	-0.91 E-4

**Table 4:** Expansion of  $v^{(2)}$  (Surface 1)

$x_1$	$x_2$	$x_3$	$v^{(2)}$	$E_5^{(2)}$	$E_6^{(2)}$
1.0	1.5	2.0	0.5509 E-1	0.33 E- 9	0.23 E-10
1.5	0.0	0.0	0.1008 E+0	0.88 E- 8	-0.99 E- 9
3.0	0.0	0.0	0.4874 E-1	0.73 E-10	-0.39 E-11
0.0	2.0	0.0	0.7352 E-1	-0.24 E- 8	-0.20 E- 9
0.0	4.0	0.0	0.3606 E-1	-0.17 E-10	-0.10 E-11
0.0	0.0	2.5	0.5788 E-1	-0.59 E- 9	-0.35 E-11
0.0	0.0	5.0	0.2862 E-1	-0.35 E-11	-0.51 E-13

**Table 5:** Expansion of  $v^{(2)}$  (Surface 2)

$x_1$	$x_2$	$x_3$	$v^{(2)}$	$E_5^{(2)}$	$E_6^{(2)}$
1.0	1.5	2.0	0.5509 E-1	0.44 E- 8	0.12 E- 9
1.5	0.0	0.0	0.1008 E+0	-0.44 E- 9	-0.10 E- 9
3.0	0.0	0.0	0.4874 E-1	-0.27 E- 9	-0.55 E-11
0.0	2.0	0.0	0.7352 E-1	-0.98 E- 8	-0.76 E- 9
0.0	4.0	0.0	0.3606 E-1	-0.25 E- 9	-0.64 E-10
0.0	0.0	2.5	0.5788 E-1	0.97 E- 8	0.13 E- 8
0.0	0.0	5.0	0.2862 E-1	0.85 E- 9	0.98 E-10

Table 6: Expansion of  $v^{(2)}$  (Surface 3)

$x_1$	$x_2$	$x_3$	$v^{(2)}$	$E_5^{(2)}$	$E_6^{(2)}$
0.0	0.0	2.0	0.7274 E-1	-0.21 E-7	-0.48 E-8
0.0	0.0	4.0	0.3587 E-1	-0.85 E-7	-0.12 E-7
0.7	0.0	0.0	0.2325 E+0	0.14 E-6	-0.19 E-6
1.4	0.0	0.0	0.1085 E+0	-0.53 E-6	-0.15 E-6
0.0	1.0	0.0	0.1525 E+0	0.45 E-6	-0.11 E-6
0.0	2.0	0.0	0.7352 E-1	0.93 E-7	0.22 E-7
1.0	1.5	0.75	0.7782 E-1	0.92 E-7	-0.29 E-7
2.0	3.0	1.5	0.3752 E-1	-0.60 E-7	-0.20 E-7

## 10. Acknowledgements

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Particular thanks go to Dr. R. Reuter, now at IBM-Deutschland, Heidelberg, for many clarifying discussions and instructive comments. Dr. R. Reuter (1983) has also given an extended table of the first orthonormal systems of homogeneous harmonic polynomials in higher dimensions.

**Appendix 1:**

maximal linearly independent systems of homogeneous harmonic polynomials for  $n = 3$  through  $n = 10$  (cf. chapter 6).

N = 4			
X1	X2	X3	X1 X2 X3
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	2
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	3
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	4
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	5
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	6
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	7
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	8
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

N = 3			
X1	X2	X3	X1 X2 X3
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	2
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	3
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	4
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	5
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	6
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0
1	1	1	7
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0





N = 6			
X1 X2 X3	X1 X2 X3	X1 X2 X3	X1 X2 X3
MC(6, 1) =	1 7 1 0	-1 0 0 0	-7 0 2 6
	7 5 3 0	20 6 2 0	
	-7 3 5 0	-70 4 4 0	MC(6, 16) =
	-1 1 7 0	20 2 6 0	7 4 1 1
		-1 0 0 0	-35 4 1 3
MC(6, 2) =			21 2 1 5
	1 7 0 1	MC(6, 10) =	-1 0 1 7
	-21 5 2 1	-35 4 3 1	
	35 3 4 1	21 2 5 1	MC(6, 17) =
	-7 1 6 1	-1 0 7 1	-1 0 0 0
MC(6, 3) =			
	3 7 1 0	MC(6, 11) =	-1 0 0 0
	-14 5 3 0	21 6 2 0	
	-21 5 1 2	7 6 0 2	
	7 3 5 0	-35 4 4 0	
	70 3 3 2	-105 4 2 2	
	-21 1 5 2	7 2 6 0	
MC(6, 4) =			
	3 7 0 1	MC(6, 12) =	-7 0 6 2
	-42 5 2 1	MC(6, 12) =	
	-7 5 0 3	3 4 1 1	
	35 3 4 1	-10 4 3 1	
	70 3 2 3	-5 4 1 3	
	-35 1 4 3	3 2 5 1	
MC(6, 5) =			
	3 7 1 0	MC(6, 13) =	-1 0 5 3
	-7 5 3 0	MC(6, 13) =	
	-42 5 1 2	-3 0 0 0	
	70 3 3 2	42 6 2 0	
	35 3 1 4	-35 4 4 0	
	-35 1 3 4	-420 4 2 2	
MC(6, 6) =			
	3 7 0 1	MC(6, 14) =	-35 0 4 4
	-21 5 2 1	210 2 4 2	
	-14 5 0 3	210 2 2 4	
	70 3 2 3	-35 0 4 4	
	7 3 0 5	MC(6, 14) =	
	-21 1 2 5	3 4 1 1	
MC(6, 7) =			
	1 7 1 0	MC(6, 15) =	-5 4 3 1
	-21 5 1 2	10 2 3 3	
	35 3 1 4	3 2 1 5	
	-7 1 1 4	-1 0 3 5	
MC(6, 8) =			
	1 7 0 1	MC(6, 15) =	-1 0 0 0
	-7 5 0 3	7 6 2 0	
	-42 5 2 1	21 6 0 2	
	70 3 3 2	-105 4 2 2	
	35 3 0 5	-35 4 0 4	
	-21 1 0 7	105 2 2 4	
MC(6, 9) =			
	1 7 0 1	MC(6, 15) =	-1 0 0 0
	-7 5 0 3	7 6 2 0	
	-42 5 2 1	21 6 0 2	
	70 3 3 2	-105 4 2 2	
	35 3 0 5	-35 4 0 4	
	-21 1 0 7	105 2 2 4	

N = 7			
X1 X2 X3	X1 X2 X3	X1 X2 X3	X1 X2 X3
MC(7, 1) =	1 7 0 0	3 4 1 0	
	-21 5 2 0	-10 4 3 0	
	35 3 4 0	-15 4 1 2	
	-7 1 6 0	3 2 5 0	
		30 2 3 2	
		-3 0 5 2	
MC(7, 2) =			
	3 5 1 1	MC(7, 11) =	
	10 3 3 1	3 4 0 1	
	-3 1 5 1	-30 4 2 1	
MC(7, 3) =			
	3 7 0 0	15 4 0 3	
	-42 5 2 0	15 2 4 1	
	-21 5 0 2	30 2 2 3	
	35 3 4 0	-5 0 4 3	
	210 3 2 2	MC(7, 12) =	
	-105 1 4 2	3 4 1 0	
MC(7, 4) =			
	3 5 1 1	-5 4 3 0	
	5 3 3 1	30 4 1 2	
	5 3 1 3	15 2 1 4	
	-5 1 3 3	-5 0 3 4	
MC(7, 5) =			
	3 7 0 0	MC(7, 13) =	
	-21 5 2 0	3 4 0 1	
	-42 5 0 2	-15 4 2 1	
	210 3 2 2	30 2 2 3	
	35 3 0 4	3 2 0 5	
	-105 1 2 4	-3 0 2 5	
MC(7, 6) =			
	1 4 1 0	MC(7, 14) =	
	-15 4 1 2	1 4 1 0	
	10 3 1 3	15 2 1 4	
	-3 1 1 5	-1 0 1 4	
MC(7, 7) =			
	1 7 0 0	MC(7, 15) =	
	-21 5 0 2	7 4 0 1	
	35 3 0 4	-35 4 0 3	
	-7 1 0 6	21 2 0 5	
MC(7, 8) =			
	7 4 1 0	-1 0 0 7	
	-35 4 3 0		
	21 2 5 0		
	-1 0 7 0		
MC(7, 9) =			
	1 4 0 1		
	-15 4 2 1		
	15 2 4 1		
	-1 0 4 1		
MC(7, 10) =			
	1 4 0 1		
	-15 4 2 1		
	15 2 4 1		
	-1 0 4 1		

N = 10			
X1 X2 X3	X1 X2 X3	X1 X2 X3	X1 X2 X3
MC(10, 1) =	-5 9 1 0	-1 9 1 0	252 2 4 2
	40 7 3 0	27 7 1 2	-9 0 0 2
	-126 5 5 0	-43 5 3 2	MC(10, 14) =
	40 3 7 0	-43 5 1 4	-3 0 1 1
	-5 1 9 0	105 3 3 4	21 4 3 1
		21 3 1 4	7 4 1 3
MC(10, 2) =	-1 9 0 1	-21 1 3 6	-35 4 5 1
	34 7 2 1	MC(10, 8) =	3 2 7 1
	-124 5 4 1	-5 9 0 1	21 2 5 3
	84 3 4 1	45 7 2 1	-1 0 7 3
	-9 1 0 1	45 7 0 3	
		-315 5 2 3	MC(10, 15) =
MC(10, 3) =	-5 9 1 0	-43 5 0 5	1 10 0 0
	45 7 3 0	315 3 2 5	-27 0 2 0
	45 7 1 2	15 3 0 7	-10 0 0 2
	-43 5 5 0	-45 1 2 7	43 4 0 0
	-315 5 3 2	MC(10, 9) =	378 4 2 2
	13 3 7 0	-1 9 1 0	31 4 0 0
	315 3 5 2	34 7 1 2	-21 4 0 0
	-45 1 7 2	-126 5 1 4	-430 4 2 4
		84 3 1 4	126 2 4 4
MC(10, 4) =	-1 9 0 1	-9 1 1 0	315 2 4 4
	27 7 2 1	MC(10, 10) =	-21 0 4 4
	3 7 0 3	-5 9 0 1	
	-43 5 4 1	40 7 0 3	MC(10, 16) =
	-43 5 2 3	-126 5 0 5	-45 0 1 1
	21 3 4 1	40 3 0 7	210 4 3 1
	105 3 4 3	-5 1 0 9	-105 4 5 1
	-21 1 4 3	MC(10, 11) =	-700 4 3 3
		-105 4 1 5	
MC(10, 5) =	-5 9 1 0	1 10 0 0	210 2 5 3
	30 7 3 0	-45 0 2 0	210 2 3 5
	90 7 1 2	210 4 4 0	
	-21 5 5 0	-210 4 0 0	MC(10, 17) =
	-420 5 3 2	-1 0 10 0	1 10 0 0
	-105 5 1 4	MC(10, 12) =	-10 0 2 0
	210 3 5 2	-9 0 1 1	-27 0 0 2
	330 3 3 4	84 4 3 1	21 4 0 0
	-105 1 5 4	-126 4 5 1	370 4 2 2
		34 2 7 1	43 4 0 4
MC(10, 6) =	-5 9 0 1	-1 0 9 1	-315 4 4 2
	70 7 2 1	MC(10, 13) =	-430 4 2 4
	30 7 0 3	-36 0 2 0	315 2 4 4
	-105 5 4 1	-9 0 0 2	126 2 4 6
	-420 5 2 3	-36 0 0 0	-21 0 4 6
	330 3 4 3	126 4 0 0	
	210 3 2 5	252 4 2 2	MC(10, 18) =
	-105 1 4 5	-84 4 6 0	-3 0 1 1
		-430 4 4 2	7 4 3 1
MC(10, 7) =	-105 1 4 5	-430 4 4 2	21 4 1 3
		-35 4 1 4	-35 4 3 3

N = 9			
X1 X2 X3	X1 X2 X3	X1 X2 X3	X1 X2 X3
MC(9, 1) =	21 3 0 4	42 2 5 2	
	-43 1 2 6	70 2 3 4	
	34 7 2 0	MC(9, 8) =	-7 0 5 4
	-126 5 4 0	MC(9, 15) =	
	84 3 4 0	1 7 1 1	-3 0 0 1
	-9 1 0 0	-7 5 1 3	42 4 2 1
		7 3 1 5	14 4 0 3
MC(9, 2) =	-1 1 1 7	-35 4 4 1	-140 4 2 3
	7 1 1 1	MC(9, 9) =	-7 4 0 5
	-7 5 3 1	-1 9 0 0	70 2 4 3
	7 3 5 1	34 7 0 2	42 2 2 5
	-1 1 7 1	-126 5 0 4	-7 0 4 5
		84 3 0 4	
MC(9, 3) =	-1 9 0 0	-9 1 0 0	MC(9, 16) =
	27 7 2 0	MC(9, 10) =	-3 0 1 0
	9 7 0 2	-9 0 1 0	43 4 1 2
	-43 5 4 0	84 4 3 0	-105 4 3 2
	-105 3 2 2	-126 4 5 0	-105 4 1 4
	21 3 6 0	34 2 7 0	105 2 3 4
	315 3 4 2	-1 0 9 0	21 2 1 4
	-43 1 4 2	MC(9, 11) =	-7 0 3 4
		-1 0 0 1	
MC(9, 4) =	7 1 1 1	20 4 2 1	MC(9, 17) =
	-42 5 3 1	-70 4 4 1	-1 0 0 1
	-21 5 1 3	20 2 4 1	7 4 2 1
	70 3 5 1	-1 0 0 1	-7 4 0 3
	-21 1 5 3	MC(9, 12) =	-35 4 2 3
		-1 0 1 0	-7 4 0 5
MC(9, 5) =	-1 9 0 0	21 2 2 5	1 2 0 7
	10 7 2 0	7 4 3 0	-1 0 2 7
	10 7 0 2	-7 4 1 2	MC(9, 18) =
	-21 5 4 0	-25 4 5 0	-1 0 1 0
	-252 5 2 2	1 2 2 5	20 4 1 2
	-21 5 0 4	21 2 5 2	-70 4 1 4
	210 3 4 2	-1 0 7 2	70 2 1 4
	210 3 2 6	MC(9, 13) =	-1 0 1 0
	-105 1 4 4	-3 0 0 1	MC(9, 19) =
		63 4 2 1	-9 0 0 1
MC(9, 6) =	7 1 1 1	-105 4 4 1	84 0 0 3
	-21 5 3 1	-105 4 2 3	-126 4 0 5
	-42 5 1 3	21 2 4 1	34 2 0 7
	70 3 3 3	105 2 4 3	-1 0 0 9
	-21 1 3 3	-7 0 4 2	
		MC(9, 14) =	-3 0 1 0
MC(9, 7) =	-1 9 0 0	-3 0 1 0	
	9 7 2 0	14 4 3 0	
	27 7 0 2	42 4 1 2	
	-105 5 2 2	-7 4 5 0	
	-43 5 0 4	-140 4 3 2	
	315 3 2 4	-35 4 1 4	

M 11			
X1	X2	X3	X1 X2 X3
M(11, 1) =	-21	5 1 5	-11 10 1 0
	-35	7 0 3	-145 0 3 0
	-330	7 4 0	-462 4 5 0
	-462	5 4 0	-330 4 7 0
	165	3 0 0	55 2 9 0
	-11	1 10 0	-1 0 11 0
M(11, 2) =	-110	9 2 0	M(11, 13) =
	-165	7 4 0	-45 0 2 1
	165	7 4 0	210 4 4 1
	2970	7 2 2	-45 2 8 1
	-126	5 5 1	-210 4 4 1
	-495	7 0 4	-45 2 8 1
	-3465	5 4 2	-1 0 10 1
	-6930	5 2 4	M(11, 14) =
	-231	5 0 4	5 10 1 0
	5775	3 4 4	-40 0 3 0
	2310	3 2 6	-45 0 1 2
	-1155	1 4 6	126 4 5 0
M(11, 3) =	-220	9 2 0	420 4 3 2
	-55	9 0 2	-60 4 7 0
	990	7 4 0	-60 4 5 2
	1980	7 2 2	5 2 9 0
	-924	5 4 0	180 2 7 2
	-6930	5 4 2	M(11, 15) =
	165	3 0 0	-15 1 3 7
	4620	3 4 2	-5 10 0 1
	-495	1 0 2	-36 0 2 1
M(11, 4) =	-5	9 1 1	-3 0 0 3
	45	7 3 1	126 4 4 1
	15	7 1 3	84 4 2 3
	-43	5 5 1	-84 4 4 1
	-105	5 3 3	-210 4 4 3
	15	3 7 1	9 2 0 1
	185	3 5 3	84 2 4 3
	-15	1 7 3	-3 0 0 3
M(11, 5) =	5	11 0 0	M(11, 16) =
	-165	9 2 0	5 10 1 0
	-110	9 0 2	-45 0 3 0
	495	7 4 0	-90 0 1 2
	2970	7 2 2	43 4 5 0
	165	7 0 4	430 4 3 2
	-231	5 4 0	105 4 1 4
	-4950	5 4 2	-15 4 7 0
	-3465	5 2 4	-5 1 1 9
	2310	3 4 2	M(11, 11) =
	5775	3 4 4	1 11 0 0
	-1155	1 4 4	-55 9 0 2
M(11, 6) =	-5	9 1 1	330 7 6 4
	30	7 3 1	-462 5 6 4
	30	7 1 3	165 3 0 0
	-21	5 5 1	-11 1 0 10
	-140	5 3 3	M(11, 12) =
			-30 0 0 3

M 10			
X1	X2	X3	X1 X2 X3
M(10, 15) =	-21	4 1 3	
	21	2 3 5	
	3	2 1 7	
	-1	0 3 7	
	1	10 0 0	
	-9	0 2 0	
	-36	0 0 2	
	252	4 2 2	
	126	4 0 4	
	-630	4 2 4	
	-84	4 0 4	
	252	2 2 6	
	9	2 0 6	
	-9	0 2 0	
M(10, 20) =	-9	0 1 1	
	84	4 1 3	
	-126	4 1 5	
	36	2 1 7	
	-1	0 1 9	
M(10, 21) =	1	10 0 0	
	-45	0 0 2	
	210	4 0 4	
	-210	4 0 4	
	45	2 0 0	
	-1	0 0 10	

**Appendix 2:**

orthonormal systems of homogeneous harmonic polynomials for  $n = 3$  through  $n = 10$  (cf. chapter 7).

N = 4	
2 3 0 0 0	2 3 0 0 0
U(4, 0) :	
1 0 0 3 1 0	
-1 0 0 1 3 0	
1 2 1	
U(4, 3) :	
1 0 0 3 0 1	
-1 0 0 1 2 1	
1 3 1	
U(4, 3) :	
1 0 0 3 1 2	
1 0 0 1 3 0	
-1 1 1 1 1 2	
7 2 1	
U(4, 0) :	
1 0 1 3 0 1	
1 0 1 1 2 1	
-1 2 0 1 0 3	
7 3 1	
U(4, 5) :	
-1 0 0 0 0 0	
1 1 1 2 0 0	
-1 0 0 0 0 0	
1 4 1	
U(4, 0) :	
1 0 1 2 1 1	
-1 0 0 0 3 1	
1 3 1	
U(4, 7) :	
-1 0 0 0 0 0	
1 1 1 2 0 2	
1 0 0 0 0 0	
-1 1 1 0 2 2	
7 4 1	
U(4, 0) :	
1 0 1 2 1 1	
1 0 1 0 3 1	
-1 2 0 0 1 3	
7 3 1	
U(4, 7) :	
-1 0 1 0 0 0	
-1 1 1 2 0 0	
1 3 1 2 0 2	
-1 0 1 0 0 0	
1 3 1 0 2 2	
-1 3 0 0 0 4	
25 1	

N = 3	
2 3 0 0 0	2 3 0 0 0
U(3, 0) :	
1 0 0 3 0 0	
-1 0 0 1 2 0	
1 3 1	
U(3, 3) :	
1 0 0 1 1 1	
1 0 0	
U(3, 3) :	
1 0 0 3 0 0	
1 0 0 1 2 0	
-1 2 0 1 0 2	
5 3 0	
U(3, 0) :	
1 0 1 2 1 0	
-1 0 0 0 3 0	
1 3 1	
U(3, 5) :	
1 0 0 2 0 1	
-1 0 0 0 2 1	
1 2 0	
U(3, 0) :	
1 0 0 2 1 0	
1 0 0 0 3 0	
-1 2 0 0 1 2	
5 3 0	
U(3, 7) :	
1 0 1 2 0 1	
1 0 1 0 2 1	
-1 1 0 0 0 3	
5 2 1	

n = 6		
2 3 5 11 12 13	2 3 5 11 12 13	2 3 5 11 12 13
0(6, 1) =	-1 0 1 0 5 1 0	1 0 0 0 0 0 0
	-1 1 0 1 3 0	1 0 0 1 2 0
	-1 0 0 1 1 3 0	-1 0 1 1 2 0
	1 7 2 1	-1 0 1 1 2 0
0(6, 2) =		-1 0 1 1 2 0
	-1 0 0 0 3 0 1	1 2 1 2 2 2
	-1 1 0 1 3 1	1 0 0 0 0 0
	-1 0 0 1 1 4 1	-1 1 0 1 4 2
	1 7 1 1	10 0 1 1
0(6, 3) =		
	-1 0 0 0 5 1 0	-1 0 2 0 4 1 1
	1 1 0 1 3 1 2	-1 1 0 2 3 1
	1 0 0 0 1 5 0	1 1 0 2 1 3
	-1 1 0 1 1 3 2	1 0 1 0 5 1
	11 6 1 1	-1 2 0 0 3 3
0(6, 4) =		11 7 2 0
	-1 0 1 0 3 0 1	1 0 0 0 0 0
	1 1 1 0 3 2 1	1 0 0 0 2 0
	1 3 0 0 3 0 3	-1 0 0 0 6 2
	1 0 2 0 1 4 1	-1 0 0 2 0 0
	-1 1 1 0 1 2 3	1 0 0 2 0 4
	11 7 2 0	-1 0 0 0 0 0
0(6, 5) =		
	-1 0 0 0 5 1 0	1 0 0 0 0 0
	1 4 0 0 3 1 2	1 0 0 0 0 0
	-1 0 0 0 1 5 0	1 0 0 0 0 0
	1 4 0 0 1 3 2	-1 0 1 4 1 1
	-1 4 0 0 1 1 4	1 2 0 1 2 1 3
	11 7 2 0	-1 0 1 0 5 1
0(6, 6) =		1 2 0 1 0 3 3
	-1 0 0 1 3 0 1	-1 3 0 0 1 3
	-1 1 0 1 3 2 1	11 6 2 1
	1 2 0 1 3 0 3	
	-1 0 0 1 1 4 1	
	1 2 0 1 1 2 3	
	-1 3 0 0 1 0 3	
	11 6 2 1	
0(6, 7) =		
	1 0 0 0 6 0 0	1 0 0 1 6 0 0
	-1 0 1 1 4 2 0	1 0 1 1 2 0
	1 0 1 1 2 0 0	-1 1 2 1 4 0 2
	-1 0 0 0 0 6 0	1 0 1 1 2 4 0
	1 9 2 1	-1 2 2 1 2 2 2
		1 3 1 1 2 0 4
		1 0 0 1 0 4 0
		-1 3 2 1 0 4 2
		1 3 1 1 0 2 4
		-1 4 0 0 0 4
		77 0 3 1
0(6, 8) =		
	-1 0 0 1 4 1 1	
	-1 0 1 2 3 1	
	-1 0 0 0 5 1	
	1 7 1 1	

n = 5		
2 3 5 11 12 13	2 3 5 11 12 13	2 3 5 11 12 13
0(5, 1) =	-1 0 0 0 5 0 0	1 0 0 0 4 1
	1 1 0 1 3 2 0	-1 1 0 0 2 3
	-1 0 0 1 1 4 0	1 0 2 0
	1 7 1 1	
0(5, 2) =		
	-1 0 0 0 4 1 0	
	-1 1 0 2 3 0	
	1 2 1 0 2 1 2	
	-1 0 0 0 5 0	
	1 2 1 0 0 3 2	
	-1 3 0 0 1 4	
0(5, 3) =		
	-1 0 0 0 5 0 0	
	1 1 0 0 3 2 0	
	1 3 0 0 3 0 2	
	-1 0 0 1 4 0	
	1 2 1 0 1 2 2	
	-1 3 0 0 1 0 4	
	7 6 2 0	
0(5, 4) =		
	-1 0 0 1 4 1 0	
	1 1 0 1 2 3 0	
	-1 0 0 0 5 0	
	1 7 1 1	
0(5, 5) =		
	-1 0 0 0 5 0 0	
	-1 1 0 0 3 2 0	
	1 2 1 0 3 0 2	
	-1 0 0 1 4 0	
	1 2 1 0 1 2 2	
	-1 3 0 0 1 0 4	
	7 6 2 0	
0(5, 6) =		
	-1 0 0 1 4 1 0	
	1 1 0 1 2 3 0	
	-1 0 0 0 5 0	
	1 7 1 1	
0(5, 7) =		
	-1 0 0 0 4 0 1	
	1 1 1 0 2 1 1	
	-1 0 0 0 4 1	
	1 6 1 0	
0(5, 8) =		
	-1 0 1 0 4 1 0	
	-1 1 0 0 2 3 0	
	1 3 1 0 2 1 2	
	1 0 0 0 5 0	
	-1 3 0 0 3 2	
	1 7 1 0	
0(5, 9) =		
	-1 0 0 0 4 0 1	
	-1 1 0 0 2 3 1	
	1 3 1 0 2 1 2	
	1 0 0 0 5 0	
	-1 3 0 0 3 2	
	1 7 1 0	







0 = 0			
2 3 5 7 11 12 13	2 3 5 7 11 12 13	2 3 5 7 11 12 13	2 3 5 7 11 12 13
1 4 1 0 0 2 2 5	-1 2 1 1 6 2 1	1 2 0 1 1 7 0 3	-1 7 1 0 0 1 1 8
-1 0 0 0 0 0 1	1 5 1 1 4 0 3	1 1 1 1 5 4 1	4199 16 3 1 1
1 3 0 0 0 0 4 5	-1 1 3 1 1 4 0 1	-1 2 2 1 5 2 3	ε(10, 10) =
17 12 2 1 0	1 5 2 1 1 4 2 3	-1 3 0 1 5 0 5	
ε(9, 10) =	-1 5 3 0 1 4 0 5	-1 2 0 2 1 3 4 3	-1 0 2 0 1 0 1
-1 0 1 0 0 0 1 0	-1 2 2 1 1 2 6 1	1 4 1 1 3 2 5	-1 2 0 1 7 2 1
-1 3 0 0 0 4 3 0	1 5 2 1 1 2 6 3	-1 0 1 2 0 1 0 1	1 3 1 1 7 0 3
1 2 3 0 0 4 1 2	-1 6 3 0 1 2 2 5	1 2 0 2 1 1 6 3	-1 1 3 0 1 5 6 1
-1 1 0 0 4 5 0	-1 0 2 1 1 0 0 1	323 13 3 2 1	-1 3 2 1 5 2 3
1 2 2 1 0 4 3 2	1 0 2 1 1 0 0 1	ε(10, 7) =	-1 2 0 1 3 6 1
-1 3 2 1 0 4 1 4	-1 5 3 0 1 0 6 5	-1 9 0 0 9 1 0	-1 6 2 0 1 3 2 5
1 2 2 0 2 5 2	1 0 2 0 0 2 7	-1 1 0 0 7 3 0	-1 0 2 0 1 0 1
-1 4 1 1 0 2 3 4	-1 7 0 0 0 0 9	1 1 1 0 1 7 1 2	1 3 1 1 1 6 3
1 6 1 0 2 1 4	263 16 4 2 1	-1 3 1 0 1 5 1 4	-1 5 2 0 1 4 5
1 0 0 0 0 9 0		-1 1 0 0 3 7 0	-1 7 2 0 0 1 2 7
-1 2 2 0 0 7 2		-1 1 0 1 3 5 2	-1 7 0 0 0 1 0 9
1 3 1 1 0 0 3 4		1 4 0 0 1 3 1 6	4199 16 4 1 1
ε(9, 10) =		1 0 0 0 1 3 1 6	
221 13 3 1 0		-1 1 0 0 6 1 9 0	ε(10, 11) =
ε(9, 17) =		-1 1 0 0 1 7 2	
-1 0 0 0 1 0 0 1		-1 3 1 0 1 5 1 4	1 0 0 0 10 0 0
-1 1 0 0 1 6 2 1		323 10 3 1 1	-1 0 2 1 0 8 2 0
1 1 0 1 1 6 3		ε(10, 0) =	1 1 1 1 1 4 0
1 1 0 1 1 2 3		-1 0 0 1 0 0 1	-1 1 1 1 1 4 0
-1 0 0 0 1 4 0 5		1 2 1 0 1 7 0 3	1 0 2 1 0 2 8 0
-1 0 0 1 2 6 1		1 1 0 1 5 4 1	-1 0 0 0 0 10 0
1 5 0 0 2 0 7		-1 2 1 0 1 5 2 3	1 1 7 4 2 1
1 0 0 1 0 0 1		-1 3 0 1 5 0 5	ε(10, 12) =
-1 0 1 0 0 3		-1 3 0 1 3 6 1	-1 0 2 0 0 1 1
-1 4 0 1 0 4 3		-1 2 1 1 3 4 3	-1 2 0 1 4 5 1
-1 5 0 0 0 2 7		1 4 0 1 3 2 5	1 2 2 0 2 7 1
ε(9, 10) =		1 6 0 0 3 0 7	-1 9 0 0 0 6 1
221 11 2 1 1		-1 0 1 0 1 0 1	1 15 4 1 1
ε(9, 10) =		-1 2 2 0 1 1 6 3	ε(10, 13) =
-1 0 0 0 1 0 1 0		3 2 0 1 1 4 5	
-1 3 0 1 1 6 3		-1 6 1 0 0 1 2 7	1 0 0 0 10 0 0
-1 1 0 1 4 5 0		323 13 3 1 1	-1 0 3 0 9 4 2 0
1 3 1 1 4 3 2		ε(10, 9) =	-1 1 2 0 9 0 2
-1 5 0 1 6 1 4		-1 0 0 1 9 1 0	1 1 1 0 1 6 0
-1 2 0 0 1 2 7 0		-1 2 0 0 1 7 3 0	1 3 2 5 1 6 2 2
1 3 1 1 2 5 2		1 4 0 1 7 1 2	1 1 0 1 6 6 0
-1 6 1 1 2 3 4		-1 1 0 1 5 5 0	-1 2 2 1 4 6 2
1 7 0 0 1 2 1 6		1 4 2 0 1 5 3 2	-1 0 3 0 2 0 0
-1 0 0 1 0 9 0		-1 6 0 1 1 3 4	1 3 2 0 1 2 4 2
1 3 0 1 1 0 7 2		-1 2 0 0 1 9 0	1 3 0 0 0 10 0
-1 5 0 1 1 0 5 4		1 5 2 0 0 1 7 2	-1 1 2 0 0 9 2
1 7 0 0 1 0 3 6		-1 3 0 0 1 1 5 4	
-1 7 0 0 0 1 0		323 15 3 1 1	ε(10, 14) =
263 14 2 1 1		-1 0 0 1 1 9 0	-1 0 1 0 1 0 1 1
ε(9, 17) =		-1 4 0 1 1 7 2	1 2 1 2 1 5 1 1
-1 0 2 1 1 0 0 1		1 0 3 1 1 5 4	1 4 9 1 5 1 3
		1 0 3 1 1 5 4	1 1 5 1 6 5 1



**Glossary of Notations:**

$$\left[ \frac{n}{2} \right] = \begin{cases} n/2 & \text{for even } n \\ (n-1)/2 & \text{for odd } n \end{cases}$$

$$\mu = 0, (1), k \quad \mu = 0, 1, \dots, k-1, k$$

$$\mu = 0, (2), 2k \quad \mu = 0, 2, \dots, 2k-2, 2k$$

$$[\alpha] = \alpha_1 + \alpha_2 + \alpha_3$$

$$\alpha! = \alpha_1! \alpha_2! \alpha_3!$$

( $\alpha$ : multiindex)

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$$

$$(\nabla_x)^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial x_2} \right)^{\alpha_2} \left( \frac{\partial}{\partial x_3} \right)^{\alpha_3}$$

$$\sum_{[\alpha]=n} C_\alpha = \alpha_1 + \alpha_2 + \alpha_3 = n \quad C_{\alpha_1 \alpha_2 \alpha_3}$$

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